# Transformation-Based Constraint-Guided Generalised Modus Ponens

Michaël Blot, Marie-Jeanne Lesot, Marcin Detyniecki
Sorbonne Universités, LIP6,
UPMC Univ Paris 06, CNRS, UMR 7606
F-75005, Paris, France
Email: {michael.blot,marie-jeanne.lesot,marcin.detyniecki}@lip6.fr

***Abstract*—Generalised Modus Ponens (GMP) allows to perform logical inference in the case where an observation partially matches the premise of an implication, enriching the rule exploitation as compared to binary classical logic. This paper proposes to further enhance the rule exploitation, integrating additional constraints to guide the inference, both to reduce uncertainty in case of partial match and to perform inference in the case of an observation disjoint from the rule premise. These constraints are expressed as logical predicates derived from properties that characterise the observation, in an absolute way or relatively to the rule premise. An extension of GMP is proposed, to take into account the constraints, based on transformation operations applied to the fuzzy sets involved in the rule and the observation. An instantiation to a GMP preserving graduality and ambiguity is established and its validity is proven.**

## I. Introduction

Given an implication $A_0 \Rightarrow B_0$ and an observation $A$, classical logic only allows to infer new knowledge in the case where the observation perfectly matches the rule premise, $A = A_0$. The Generalised Modus Ponens (GMP) [1] enriches the rule exploitation, allowing to perform logical inference even when $A$ differs from $A_0$, to a certain extent. In most cases, the imperfect match between $A$ and $A_0$ induces a result with uncertainty, expressing the fact that other values, not contained in the rule, cannot be excluded.

This paper proposes to extend GMP, so as to exploit additional constraints to guide the inference in this case of partial match: in particular it aims at getting more certain results, that better capture the relation between the observation and the premise. These additional constraints can for instance express the preservation of some properties, e.g. the ambiguity level of the observation, or specific behaviours, e.g. a monotonous relationship between the observation and the premise.

The introduction of these constraints in GMP also allows to perform inference when the observation does not match the rule premise at all: although the rule is not to be triggered in such a case, it can be considered as providing information allowing to process the observation and to infer new knowledge from it.

The proposed GMP extension, named Transformation-based Constraint-guided Generalised Modus Ponens, T-CGMP, consists in identifying transformations that capture the relation between the observation and the rule premise, in particular in terms of the properties that appear in the considered constraints. It then transposes them to the conclusion universe.

The paper is organised as follows: Section II describes the notations and recalls the related inference principles of GMP and analogical reasoning. Section III proposes a typology of the constraints that can be considered as additional inference principles, expressed as logical predicates derived from properties that characterise the observation, in an absolute way or relatively to the rule premise. Section IV describes the proposed Transformation-based, Constraint-guided, Generalised Modus Ponens that extends GMP to take into account such additional constraints. Section V and VI focus on a specific case, where constraints of graduality and ambiguity preservation are considered: they instantiate T-CGMP to an inference principle that both satisfies monotonicity and ambiguity processing. Section VII concludes the paper and describes future works.

## II. Related Works

Given an implication $A_0 \Rightarrow B_0$ and an observation $A$, an inference scheme $F$ defines a new piece of knowledge $B$, and can be generally written as

$$B = F(A_0, B_0, A) \tag{1}$$

$$\text{such that } A = A_0 \Longrightarrow B = B_0 \tag{2}$$

The constraint in Eq. (2) can also be written as imposing that $F$ is such that $F(A_0, B_0, A_0) = B_0$: if the observation equals the rule premise, the inferred result equals the rule conclusion.

This section briefly recalls the notations of the Generalised Modus Ponens, some of its variants that take into account additional constraints and the related inference principle of analogical reasoning.

### A. Generalised Modus Ponens

Denoting $\mathcal{X}$ and $\mathcal{Y}$ the premise and conclusion universes, $A_0, A$ two fuzzy sets defined on $\mathcal{X}$, $B_0$ a fuzzy set defined on $\mathcal{Y}$ and using the same symbol for a fuzzy set and its membership function, the Generalised Modus Ponens [1] implements the above general inference scheme as

$$\begin{aligned} & B = GMP(A, A_0 \Rightarrow B_0) \\ \Leftrightarrow \quad & \forall y \in \mathcal{Y},\ B(y) = \sup_{x \in \mathcal{X}} \top(A(x), \mathcal{I}(A_0(x), B_0(y))) \end{aligned} \tag{3}$$

where $\top$ is a triangular norm and $\mathcal{I}$ a fuzzy implication operator, jointly chosen so that the constraint defined in Eq. (2) is satisfied.

As a result, and as opposed to the classic Modus Ponens, GMP allows to perform informative inference even when the observation does not exactly match the rule premise. In most cases, the imperfect match between $A$ and $A_0$ induces a result with uncertainty, expressing the fact that other values, not contained in the rule, cannot be excluded. This behaviour is illustrated on the two left graphs of Figure 1: the leftmost one shows the premise $A_0$ in red and a partially matching observation $A$ in blue. The middle graph shows the rule conclusion $B_0$ in red and, in blue, the inferred $B = GMP(A, A_0 \Rightarrow B_0)$, as defined in Eq. (3), considering the Łukasiewicz implication $\mathcal{I}(u, v) = \max(1 - u, v)$ and the Łukasiewicz t-norm $\top(u, v) = \max(0, u+v-1)$. The GMP result is indeed similar to the rule conclusion, but includes an uncertainty level for all values of the universe.

In the case where $A_0$ and $A$ have disjoint supports, i.e. when the observation does not match at all the rule premise, in most combinations of implication and t-norm operators [2], the obtained result $B$ equals the whole universe. This behaviour is illustrated on the two left graphs of Figure 2: for the observation in blue on the leftmost graph with empty intersection with the rule premise in red, the GMP inferred B (in blue on the middle graph) has membership degree 1 for all values of the universe. This behaviour relies on an uncertainty-based reading of fuzzy sets: all values of the universe are equally, and totally, possible. This is a reasonable choice considering that, in this case, the rule actually does not apply.

The Mamdani inference system [3], [4] replaces the implication operator $\mathcal{I}$ in Eq. (3) by the $\min$ operator. The obtained result in the case of a disjoint observation is then the empty set. This choice has long been proven to be highly relevant in the domain of fuzzy control, as defining appropriate ways to determine the action to be taken after defuzzification. It has been shown that the Mamdani approach can be interpreted as a modified Generalised Modus Ponens that explicitly takes into account the rule applicability, integrating a measure of the compatibility between observation and premise when triggering it [5].

### B. Some GMP Variants

Both behaviours of GMP described above in the case of an observation disjoint from the premise are relevant, when considering that in this case, the rule actually does not apply. However, one may wish to consider that, even in this case, the rule expresses useful knowledge and that it may be exploited to perform an inference, using additional constraints.

One of the most frequent ones expresses a graduality constraint, requiring a monotonous behaviour of the inference result with respect to the rule premise (see e.g. [6]): it takes into account the relative position of the observation with respect to the premise. Considering, e.g. fuzzy sets defined on the universe $\mathcal{X} = \mathbb{R}$, it allows to get different results when the observation is on the left or on the right of the premise. It can be considered as adding, to the general form of inference expressed by Eq. (1) and (2) an additional constraint, to guide the inference.

This notion of monotononicity is understood as a global constraint across the universes, and not in terms of truth values, as is the case in gradual reasoning [7]. Several variants consider the monotonicity of a complete fuzzy inference system [8]–[11]; the case of single rules is for instance tackled by the Gradual Generalised Modus Ponens [6]. It consists in forbidding that all $x \in \mathcal{X}$ of the input universe influence on the result, as is the case for GMP, due to the aggregation $\sup_{x\in\mathcal{X}}$ used in Eq. (3): it identifies relevant subparts of the universe $\mathcal{X}$ to be taken into account when processing an observation, integrating information about the relative position of the observation with respect to the rule premise.

### C. Analogical Reasoning

The inference performed in analogical reasoning (see e.g. [12]) depends on three resemblance relations, $R_X$, $R_Y$ and $\beta$, respectively defined on $[0,1]^{\mathcal{X}} \times [0,1]^{\mathcal{X}}$, $[0,1]^{\mathcal{Y}} \times [0,1]^{\mathcal{Y}}$ and $[0,1]^{\mathcal{X}} \times [0,1]^{\mathcal{Y}}$. It can be generally written as

$$B = F(A_0, B_0, A)$$

$$\text{such that} \quad \begin{array}{l} A = A_0 \Longrightarrow B = B_0 \\ A_0 \beta B_0 \text{ and } A_0 R_X A \Longrightarrow A \beta B \text{ and } B_0 R_Y B \end{array}$$

Thus, it can be interpreted as an inference scheme of the form recalled by Eq. (1) and (2), considering an additional constraint, regarding the similarity relations between $B$ and the reference fuzzy sets $A_0$, $B_0$ and $A$.

As compared to GMP, it can be argued [13] that analogical reasoning uses the similarities $R_X$ and $R_Y$ to find $B$, the relation between $A$ and $B$ being obtained as a consequence. Reciprocally, for approximate reasoning interpreted as an analogical scheme [12], [13], the link between $A$ and $B$ is given by the GMP itself, the similarities are obtained as the consequences.

The GMP extension proposed in this paper also considers the introduction of additional constraints in the scheme recalled by Eq. (1) and (2). However, it considers a more general case than analogical reasoning, according to the typology of constraints described in the next section. The general form of the solution, used to build the inferred piece of knowledge $B$, is then described in Section IV: it exploits transformation operations, which can be interpreted as establishing resemblance relations, along the same lines as analogical reasoning, but depending on the considered constraints.

## III. Constraint Typology

This section discusses a typology of possible constraints whose preservation can be used to guide the inference performed when combining a rule of the form $A_0 \Rightarrow B_0$ with an observation $A$: such constraints can be added in the scheme recalled by Eq. (1) and (2).

The section first distinguishes between three types of characterisations of fuzzy sets, then establishes logical properties that are finally used to build the constraints.

### A. Characterisations of Fuzzy Sets

Fuzzy sets defined on a universe $\mathcal{X}$ can first be characterised using fuzzy set measures, generally written $m : [0,1]^{\mathcal{X}} \to \mathbb{R}$. They for instance include the size of their support or of their kernels, their degree of fuzziness, or other measures of fuzziness or non-specificity [14], [15].

A second type of numerical characterisations, in a relative sense, applies to couples of fuzzy sets, and not individual ones:

they capture the specificity of a fuzzy set with respect to a reference one. Such measures, globally defined as functions $m : [0,1]^{\mathcal{X}} \times [0,1]^{\mathcal{X}} \to \mathbb{R}$, for instance include similarity or inclusion measures [16].

A third type of characterisation does not reduce fuzzy sets to a numerical value, but to specific points of the universe on which they are defined, as for instance their centre of gravity, the centre of their kernel or support. These characterisations, that can also be called location measures, can be globally written as $pt : [0,1]^{\mathcal{X}} \to \mathcal{X}$.

### *B. Logical Properties of Fuzzy Sets*

The previous characterisations of fuzzy sets then lead to the definition of 4 types of logical predicates expressing properties they can satisfy.

A first type of predicates generally written $P : [0,1]^{\mathcal{X}} \to \mathbb{B}$, where $\mathbb{B}$ is the Boolean algebra, can be derived from the fuzzy set measures: for any measure $m$ and a numerical threshold $\alpha \in \mathbb{R}$, a first property can be defined as

$$P(A) \equiv m(A) \bowtie \alpha \tag{4}$$

where $\bowtie \in \{>, <, \leq, \geq, =\}$. For instance, one can define a property indicating whether the degree of fuzziness is greater than 0.8.

The fuzzy set measures that apply to couples of fuzzy sets can lead to predicates of this category, considering a reference fuzzy set: considering for instance a similarity measure $sim$, one can define $P_{A_0}(A) \equiv sim(A, A_0) \geq \alpha$; in the following, the subscript $A_0$ may be omitted.

A second type of predicates applies to couples of fuzzy sets and is generally defined as

$$P(A, B) \equiv m(A) \bowtie m(B) \tag{5}$$

where $m$ is any fuzzy set measure. This type can be illustrated by the property stating that $B$ has a greater degree of fuzziness than $A$, $P(A,B) \equiv dF(B) \geq dF(A)$. Of special interest is the case where the comparator is $=$: the property is then interpreted as a preservation property, stating that $A$ and $B$ have the same value for the considered measure. It may be generalised, using a function $f : \mathbb{R} \to \mathbb{R}$, allowing for some transformation of the value, defining $P(A, B) \equiv m(A) = f(m(B))$.

Again, predicates in this category may be built using fuzzy set measures applying to couples of fuzzy sets, considering a given reference fuzzy set. The ambiguity preservation constraint can be expressed using such a property, as detailed in Section V-B.

A third type of predicates can be defined when fuzzy sets are characterised by specific points, using predicates applying to points, in particular their distance in $\mathcal{X}$: in a relative sense, again using $\bowtie \in \{>, <, \leq, \geq, =\}$, one can for instance define

$$P(A, B) \equiv \|pt(A) - pt(B)\| \bowtie \alpha \tag{6}$$

The classical constraint of graduality can be expressed using such a property, as detailed in Section V-A.

Finally, a fourth type of predicates can be defined directly, using some logical characterisations of fuzzy sets, for instance by predicates establishing whether they are crisp or symmetrical.

### *C. Logical Constraints Imposed on Fuzzy Sets*

Based on the above defined predicates, two main types of constraints expressed on couples of fuzzy sets can be distinguished.

The first one directly defines a constraint of the form

$$C(A, B) \equiv P(A, B) \tag{7}$$

where $P$ is any predicate applying to a couple of fuzzy sets. It can be illustrated by the constraint stating that the degree of fuzziness of the result $B$ must be greater than or equal to that of $A$. It may be the case that the predicate $P$ actually depends on reference sets, in which case $C$ should indexed by the latter, e.g. $C_{A_0,B_0}(A, B)$, we omit it to ease readability.

The second category establishes relations between properties, of the form

$$C(A, B) \equiv P(A) \Longrightarrow P'(B) \tag{8}$$

where $P$ and $P'$ are two predicates. Again, potential subscripts of reference sets cab be omitted. For instance, the analogical reasoning scheme [12] can be interpreted as a constraint of this form, $C_{A_0,B_0}(A, B) \equiv sim(A, A_0) \geq \alpha \Longrightarrow sim(B, B_0) \geq \beta$, using two distinct similarity measures.

The general constraint of inference schemes defined in Eq. (2) belongs to this category, considering both $P$ and $P'$ as $=$, with the reference sets $A_0$ and $B_0$.

## IV. Proposed Transformation-Based Constraint-Guided GMP

This section describes the proposed extension of the Generalised Modus Ponens, that uses constraints as defined and represented in the previous section, so as to guide the inference: the inference issue can be written as

$$B = F(A_0, B_0, A)$$
$$\text{such that } \Gamma_{A_0,B_0}(A, B) = \bigwedge_{i=1..I} C^i_{A_0,B_0}(A, B) \tag{9}$$

where $C^i_{A_0,B_0}(A, B)$, $i = 1..I$ denote the set of considered constraints. It reduces to GMP if the constraint $\Gamma$ reduces to the classic constraint given in Eq. (2), and to analogical reasoning if a similarity-based constraint is considered.

### *A. General Form*

The proposed T-CGMP modifies the Generalised Modus Ponens, not applying it directly to the observation, $A$, the premise $A_0$ and the conclusion $B_0$, but to transformations thereof: it is defined as

$$\begin{aligned} B &= \text{T-CGMP}(A, A_0 \Rightarrow B_0) \\ \Longleftrightarrow B &= GMP(A, tA_0 \Rightarrow t'B_0) \end{aligned} \tag{10}$$

where $t$ and $t'$ are transformations, $t : [0,1]^{\mathcal{X}} \to [0,1]^{\mathcal{X}}$ and $t' : [0,1]^{\mathcal{Y}} \to [0,1]^{\mathcal{Y}}$, determined so that the constraints expressed in Eq. (9) are satisfied, as detailed in the following.

Thus, in a similar way to analogical reasoning, $t$ aims at identifying a relation between the rule premise $A_0$ and the observation $A$: in an ideal case, it should be defined such that $tA_0 = A$. However, it may be the case that such an exact transformation does not exist, therefore an approximation is considered: $t$ is required to preserve the relevant characteristics of $A$. Relevance is defined with respect to the considered constraints, i.e. with respect to $\Gamma$ as defined in Eq. (9).

The transformation $t'$ must then transpose this operation to the conclusion universe $\mathcal{Y}$, where the transposition principle also aims at satisfying the constraints defined in $\Gamma$.

### B. Implementation

Based on the definition stated in Eq. (10), the inference issue is then expressed as the identification of the transformation couple $(t, t')$, where $t$ and $t'$ depend one on each other. They cannot be determined generally and depend on the considered set of constraints $\Gamma$. Section V and VI illustrate the case of two constraints, graduality and ambiguity preservation.

The practical procedure to implement the proposed T-CGMP consists in specifying a parametrised family of transformations as well as a method to set their parameters for a given observation. It then requires to prove that the resulting transformations, when used in Eq. (10), lead to an an inference that satisfies the considered constraints $\Gamma$.

## V. Constraint Instanciation: Graduality and Ambiguity Preservation

This section describes the application of the proposed T-CGMP to two specific constraints, namely graduality and ambiguity preservation, for fuzzy sets defined on a bounded universe $\mathcal{X} \subseteq \mathbb{R}$. In this section and the following, the fuzzy subsets are assumed to be normal, with convex and compact kernels.

This section discusses the two constraints in turn and establishes the considered global constraint.

### A. Graduality Constraint

The classical graduality constraint (see, e.g. [6]) imposes a monotone behaviour of the output with respect to the input. It can be seen as interpreting the rule $A_0 \Rightarrow B_0$ in terms of a linear approximation: a small variation around $A_0$ implies a small variation around $B_0$. It moreover takes into account a directional information: two observations $A$ and $A'$, symmetrical with respect to the centre of $A_0$ kernel, then do not lead to the same conclusion, contrary to the result obtained with the GMP. Note that requiring graduality is not always relevant, but depends on the considered application.

*1) Graduality Measure:* We propose to capture the notion of graduality through a relative numerical measure $\delta : [0,1]^{\mathcal{X}} \times [0,1]^{\mathcal{X}} \to \mathbb{R}$ such that (i) $\delta(A, A') = -\delta(A', A)$ to capture a notion of relative position, and (ii) $\delta(A, A') = \delta(A, A'') + \delta(A'', A')$ to capture a notion of remoteness.

In the case where the fuzzy subsets are defined on $\mathcal{X} = \mathbb{R}$, graduality can be measured by a signed distance between characteristic points: $\delta(A, A') = pt(A) - pt(A')$. In particular, we consider the kernel centre:

$$\delta(A, A') = C_{ker}(A') - C_{ker}(A)$$

*2) Constraint Expression:* We propose to express the graduality constraint as setting an equality between $\delta(A, A_0)$ and $\delta(B, B_0)$ up to a multiplicative factor $\gamma \in \mathbb{R}$, i.e.

$$C_{grad} \equiv \delta(B, B_0) = \gamma\delta(A, A_0) \qquad (11)$$

where the dependence on $A_0$ and $B_0$ is not made explicit in the subscript.

The $\gamma$ coefficient is a hyperparameter integrating knowledge about the universe scales, and more precisely whether they are commensurable: it expresses a correspondence between the distances computed in $\mathcal{X}$ and $\mathcal{Y}$ respectively. In the case of bounded numerical universes, the absolute value of $\gamma$ can for instance be defined as

$$|\gamma| = \frac{\max(\mathcal{Y}) - \min(\mathcal{Y})}{\max(\mathcal{X}) - \min(\mathcal{X})}$$

The sign of the parameter $\gamma$ plays a major role: it indicates whether an increasing or a decreasing graduality is imposed.

In the typology described in Section III, this graduality constraint belongs to the family $C(A, B) \equiv m(A) = f(m(B))$, with a multiplicative function $f(x) = x/\gamma$ and omitted reference fuzzy sets $A_0$ and $B_0$.

### B. Ambiguity Constraint

The ambiguity constraint aims at expressing a desired behaviour of the following form: if the observation is more ambiguous than the rule premise, then the conclusion should also be more ambiguous than the rule conclusion. As graduality, requiring the preservation of ambiguity is not always relevant, but depends on the considered application.

*1) Ambiguity Measure:* There exist many measures of the ambiguity of fuzzy sets [14], [15]. We consider a basic definition related to the size of kernel, satisfying (i) $amb(\mathcal{X}) = 1$, (ii) $amb(\emptyset) = 0$ and (iii) if $|ker(A)| \leq |ker(A')|$, then $amb(A) \leq amb(A')$, and use

$$amb(A) = \frac{|ker(A)|}{|\mathcal{X}|}$$

where, for any convex bounded subset $E$ of $\mathcal{X}$, the size of $E = [e^-, e^+]$ is measured as $|E| = e^+ - e^-$.

*2) Constraint Expression:* Two principles encoding expectations about the behaviour of the inference rule regarding ambiguity can be established. First, if the observation $A$ is more ambiguous than the rule premise $A_0$, then the conclusion should also be more ambiguous than $B_0$: this behaviour can be written

$$amb(A) > amb(A_0) \Longrightarrow amb(B) > amb(B_0)$$

If, on the contrary, the observation is less ambiguous than $A_0$, the result cannot be made less ambiguous than the conclusion. Indeed, there is no way to guide the definition of a more specific fuzzy subset: there is no one-to-one correspondence between $A_0$ and $B_0$ points. If, for instance, $A \subset A_0$ (and in particular their kernels satisfy $ker(A) \subset ker(A_0)$), it can be expected that $B \subset B_0$. However, one cannot identify which points of $B_0$ should be removed to define $B$, as there is no correspondence between points in $A \backslash A_0$ and points in $B_0$. The expected behaviour when $A$ is less ambiguous than $A_0$ is therefore only the preservation of $B$ ambiguity. Formally, this constraint can be written

$$amb(A) \leq amb(A_0) \Longrightarrow amb(B) = amb(B_0)$$

In order to define a constraint capturing simultaneously these two behaviours, we propose to define

$$C_{amb} \equiv \frac{1 - amb(B)}{1 - amb(B_0)} = \min\left(1, \frac{1 - amb(A)}{1 - amb(A_0)}\right) \quad (12)$$

First, the order preservation is translated to a quotient preservation. Second, the quotient applies to the quantities $1 - amb(A)$ instead of the ambiguity value itself to avoid 0-division, in particular in the case of triangular fuzzy sets, with ambiguity 0. The special cases where $A_0 = \mathcal{X}$ or $B_0 = \mathcal{Y}$, which lead to undefined fractions in this expression, are not considered: both would correspond to uninformative rules and can be excluded without limiting the expressiveness of the constraint. Third, computing the minimum between the quotient and 1 allows to capture the desired behaviour in the case where the observation is less ambiguous than the rule premise.

### C. Instanciated Inference Issue

Using these constraints, the constrained inference principle with graduality and ambiguity preservation can thus be written

$$\begin{array}{ll} & B = F(A_0, B_0, A) \\ \text{such that} & A = A_0 \Longrightarrow B = B_0 \\ & \wedge\, C_{grad} \text{ as defined in Eq. (11)} \\ & \wedge\, C_{amb} \text{ as defined in Eq. (12)} \end{array} \quad (13)$$

## VI. T-CGMP Instantiation for Graduality and Ambiguity Preservation

This section discusses the implementation of the general principle defined in Eq. (10) to the case of the constraints given in Eq. (13): it describes the relevant transformation parametrised families and the definition of their parameters. It proves its required properties and illustrates its use on two examples.

### A. Considered Transformations

In order to implement T-CGMP with graduality and ambiguity preservation, we propose to consider two transformation families, translation and kernel extension, for fuzzy sets defined on a bounded universe of $\mathbb{R}$ with compact kernels.

Given $\tau \in \mathbb{R}$, the $\tau$-translation of a fuzzy set $A$, $t_\tau A$, is defined as

$$t_\tau A(x) = \begin{cases} A(x - \tau) & \text{if } x - \tau \in \mathcal{X} \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

Given $\Delta \in \mathbb{R}^+$, the $\Delta$-kernel extension of a fuzzy set $A$ with kernel $[a^-, a^+]$ , $T_\Delta A$, is defined as

$$T_\Delta A(x) = \begin{cases} A(x + \Delta) & \text{if } x \leq a^- \\ 1 & \text{if } x \in [a^-, a^+] \\ A(x - \Delta) & \text{if } x \geq a^+ \end{cases} \quad (15)$$

These transformations satisfy the following properties

$$\begin{array}{ll} (P1) & \delta(A, t_\tau A) = \tau \\ (P2) & amb(T_\Delta A) = \frac{2\Delta}{|\mathcal{X}|} + amb(A) \\ (P3) & \delta(T_\Delta A, A) = 0 \\ (P4) & amb(t_\tau A) = amb(A) \end{array}$$

*Proof* $(P1)$ is a direct consequence of the translation definition, likewise $(P2)$ is established from the fact that $ker(T_\Delta A) = [a^- - \Delta, a^+ + \Delta]$.

Properties $(P3)$ and $(P4)$ establish cross-effects between the transformations and the considered constraints. Informally, $(P3)$ holds because the kernel extension does not modify the kernel centre. On the other hand, $(P4)$ is proved by the fact that the translation does not impact the kernel size, only moving it globally. □

### B. Transformation Instanciations

Given the fuzzy sets $A_0$, $A$ and $B_0$, the parameters of the transformations can be set as follows.

*1) Transforming $A_0$:* The transformation applied to $A_0$ is defined as a combination of translation and kernel extension with appropriate parameters defined as:

$$\begin{array}{l} \tau = \delta(A_0, A) \\ \Delta = \dfrac{1}{2}(amb(A) - amb(A_0))|\mathcal{X}| \\ t = \begin{cases} t_\tau & \text{if } \Delta \leq 0 \\ T_\Delta t_\tau & \text{otherwise} \end{cases} \end{array} \quad (16)$$

Three properties of interest can be established

$$\begin{array}{ll} (P5) & \delta(tA_0, A) = 0 \\ (P6) & ker(A) \subseteq ker(tA_0) \\ (P7) & amb(tA_0) = \max(amb(A_0), amb(A)) \end{array}$$

*Proof of* $(P5)$ This property states that $A$ and $tA_0$ have the same kernel centre. Indeed if $\Delta \leq 0$, $t = t_\tau$, using $\delta$ transitivity and property $(P1)$, it holds that

$$\begin{aligned} \delta(tA_0, A) &= \delta(t_\tau A_0, A) \\ &= \delta(t_\tau A_0, A_0) + \delta(A_0, A) \\ &= -\tau + \tau = 0 \end{aligned}$$

Otherwise, if $\Delta > 0$, the same proof can be used after applying property $(P3)$: $\delta(tA_0, A) = \delta(T_\Delta t_\tau A_0, A) = \delta(t_\tau A_0, A)$. □

*Proof of* $(P6)$ First if $\Delta \leq 0$, $tA_0 = t_\tau A_0$ and it therefore has the same kernel centre as $A$. In addition, due to the condition on $\Delta$, $A$ kernel is smaller than that of $A_0$. Using the assumption that the considered fuzzy sets have compact kernels gives the desired result.

If, on the other hand, $\Delta > 0$, $tA_0$ extends the kernel of $t_\tau A_0$, which is defined so as to have the same centre as the kernel of $A$. Therefore $tA_0$ also has the same kernel centre as $A$; it also has the same size, as shown below in the proof of $(P7)$, which gives the desired result.

*Proof of* $(P7)$ If $\Delta \leq 0$, i.e. if $A_0$ has a higher ambiguity, $amb(tA_0) = amb(t_\tau A_0) = amb(A_0)$ using property $(P4)$. Otherwise, $amb(tA_0) = amb(T_\Delta t_\tau A_0) = amb(T_\Delta A_0) = \frac{2\Delta}{|\mathcal{X}|} + amb(A_0) = amb(A)$ due to the choice of the $\Delta$ value.

*2) Transforming $B_0$:* The transformation $t'$ is similar to $t$, adapting the parameters to the $\mathcal{Y}$ universe, i.e.

$$\tau' = \gamma\tau$$
$$\Delta' = \Delta\frac{|\mathcal{Y}|}{|\mathcal{X}|}\frac{1-amb(B_0)}{1-amb(A_0)}$$
$$t' = \begin{cases} t_{\tau'} & \text{if } \Delta \leq 0 \\ T_{\Delta'}t_{\tau'} & \text{otherwise} \end{cases} \quad (17)$$

This transformation possesses properties identical with $(P5)$, $(P6)$ and $(P7)$, transposed to $B$ and $B_0$ instead of $A$ and $A_0$.

### C. Validity Proof

This section proves that using the transformations $t$ and $t'$ defined above, in Eq. (16) and (17), the proposed T-CGMP that defines $B = GMP(A, tA_0 \Rightarrow t'B_0)$ satisfies the constraints $\Gamma$ defined in Eq. (13).

*Auxiliary Lemma* It holds that, if $U, V \in [0,1]^{\mathcal{X}}$ and $W \in [0,1]^{\mathcal{Y}}$ are normal fuzzy subsets such that $ker(U) \subseteq ker(V)$, then $Z = GMP(U, V \Rightarrow W)$ satisfies $ker(Z) = ker(W)$

*Proof* First, $ker(Z) \subseteq ker(W)$. Indeed, for $y \in \mathcal{Y}$ such that $\mu_Z(y) = 1$, the definition of GMP (see Eq. (3)) implies that there exists $x \in \mathcal{X}$ such that $\mu_U(x) = 1$ and $\mathcal{I}(\mu_V(x), \mu_W(y)) = 1$. As $ker(U) \subseteq ker(V)$, $\mu_V(x) = 1$ and therefore, due to the properties of implication functions, $\mathcal{I}(1, \mu_W(y)) = 1$ leads to $\mu_W(y) = 1$.

Reciprocally, $ker(W) \subseteq ker(Z)$: for $y \in \mathcal{Y}$ such that $\mu_W(y) = 1$, consider $x^* \in ker(U)$ ($U$ being normal has a non-empty kernel). As $ker(U) \subseteq ker(V)$, $\mu_V(x^*) = 1$, thus $\mathcal{I}(\mu_V(x^*), \mu_W(y)) = \mathcal{I}(1,1) = 1$. With $\mu_U(x^*) = 1$, applying Eq. (3) leads to $\mu_Z(x) = 1$. □

*Proof of perfect match* The first constraint in Eq. (13) can be established as follows: if $A = A_0$, then $\tau = 0$ and $\Delta = 0$, which in turn implies that $\tau' = 0$ and $\Delta' = 0$. As a consequence, $t$ and $t'$ are the identity, $B =$ T-CGMP$(A, A_0 \Rightarrow B_0) = GMP(A, tA_0 \Rightarrow t'B_0) = GMP(A, A_0 \Rightarrow B_0) = B_0$. □

*Proof of Graduality Constraint* Let us show that $C_{grad}$, as defined in Eq.(11), holds: exploiting $(P6)$ and the above established lemma, $ker(B) = ker(t'B_0)$, therefore $\delta(B, t'B_0) = 0$. Using $(P3)$ and the definition of $t'$, it implies that $\delta(t_{\tau'}B_0, B) = 0$. Therefore,

$$\begin{aligned} \delta(B_0, B) &= \delta(B_0, t_{\tau'}B_0) + \delta(t_{\tau'}B_0, B) \\ &= \tau' + 0 \\ &= \gamma\tau = \gamma\delta(A, A_0) \end{aligned} \quad \square$$

*Proof of Ambiguity Constraint* Let us show that $C_{amb}$, as defined in Eq. (12), holds: again, exploiting $(P6)$ and the above established lemma, $ker(B) = ker(t'B_0)$, therefore $amb(B) = amb(t'B_0)$.

If $\Delta \leq 0$, the right-hand side of the equality in Eq. (12) equals 1. Moreover, $t' = t_{\tau'}$ thus, using $(P4)$, $amb(t'B_0) = amb(t_{\tau'}B_0) = amb(B_0)$, establishing that the left-hand side is 1 as well, proving that $C_{amb}$ holds in this case.

If $\Delta > 0$, using $(P2)$ and $(P4)$ successively

$$\begin{aligned} amb(t'B_0) &= \frac{2\Delta'}{|\mathcal{Y}|} + amb(B_0) \\ &= \frac{2\Delta}{|\mathcal{X}|}\frac{1-amb(B_0)}{1-amb(A_0)} + amb(B_0) \\ &= (amb(A) - amb(A_0))\frac{1-amb(B_0)}{1-amb(A_0)} + amb(B_0) \end{aligned}$$

due to the choice of $\Delta$ and $\Delta'$. This equation then gives the desired result when computing the quotient involved in the constraint defined in Eq. (12). □

These proofs show that the transformations defined in Eq. (16) and (17), combined with the proposed T-CGMP defined in Eq. (10), indeed satisfy the constraints of Eq. (13). It thus provides a way to perform inference taking into account constraints imposing graduality and ambiguity constraints.

### D. Illustrative Example

Figures 1 and 2 compare the results obtained with T-CGMP and GMP: in both cases, the left graph shows the premise $A_0$ in red and the observation $A$ in blue. The middle graph shows the conclusion $B_0$ in red and, in blue, as discussed in Section II-A, the GMP results obtained with the Łukasiewicz implication and t-norm. The right graph shows the conclusion $B_0$ in red and the result $B = GMP(A, tA_0 \Rightarrow t'B_0)$, obtained using the T-CGMP instantiation described above.

Figure 1 considers a case where the observation partially matches the premise of the rule, with a slight translation to the right. Whereas GMP builds a result similar to the rule conclusion with uncertainty, T-CGMP offers a result that moves the conclusion partially to the right: the guiding principle based on the additional graduality constraint allows to dispense of the uncertainty and to obtain a conclusion that is located on the right of $B_0$, as $A$ is on the right of $A_0$.

Figure 2 illustrates the case where the observation does not match the observation: the GMP then leads to total uncertainty, as the rule cannot be triggered. The T-CGMP exploits the additional constraints of graduality and ambiguity preservation, building a transposition of the rule conclusion, both moved to the right and more ambiguous than $B_0$. Observe that the obtained solution does not have a compact support and presents a low uncertainty level. It is however much lower than the one obtained with GMP: the exploitation of the information expressed by the constraints indeed leads to a more informative result.

## VII. Conclusion and Future Works

This paper studied a variant of the fuzzy inference rule that allows to perform logical inferences even in the case where the observation does not perfectly match the implication premise: it proposed to take into account additional constraints to guide the inference and guarantee some desirable properties, such as graduality or ambiguity preservation. It introduced a general framework to express the constraint and the general form of the solution, transferring the issue of the inference definition to that of transformation definition. The proposed principles have been instantiated to the case of graduality

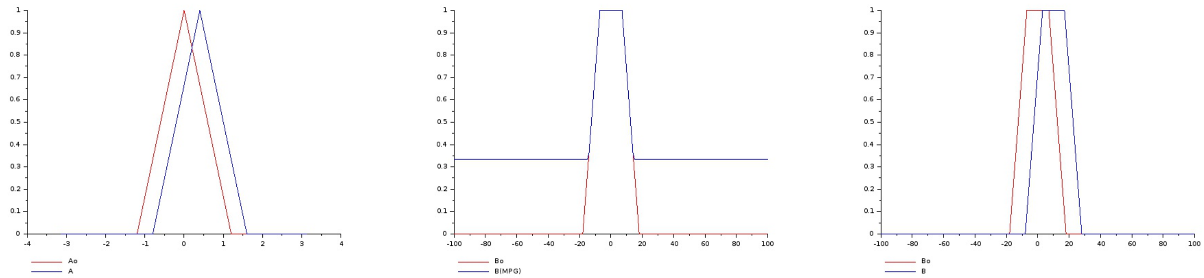


Fig. 1. Comparison between the GMP and the T-CGMP results for an observation partially matching the observation. (Left) in red, $A_0$, in blue, $A$, (Middle) in red, $B_0$, in blue, $B$ from GMP using the Łukasiewicz implication and t-norm (Right) in red, $B_0$, in blue, $B$ from T-CGMP

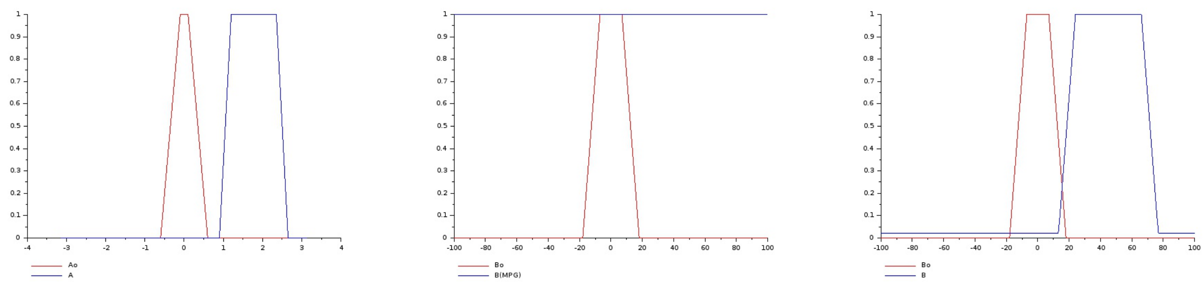


Fig. 2. Comparison between the GMP and the T-CGMP results for an observation not matching the observation and with increased ambiguity. (Left) in red, $A_0$, in blue, $A$, (Middle) in red, $B_0$, in blue, $B$ from GMP using the Łukasiewicz implication and t-norm (Right) in red, $B_0$, in blue, $B$ from T-CGMP

and ambiguity preservation, illustrating the relevance of the proposed approach.

Future works will aim at studying other instantiations of the proposed T-CGMP, for instance considering constraints on the symmetry or inclusion measures, to deal with the specific case where the observation is included in the rule premise. Another perspective will consist in exploring the case of systems with multiple rules, examining the constraints they may lead to and the required adaptations.

## References


[1] L. Zadeh, "The concept of a linguistic variable and its applications to approximate reasoning, part 1-3," *Information Science*, vol. 8-9, pp. 199–249, 301–357, 43–80, 1975.

[2] E. Trillas, "On logic and fuzzy logic," *Int. Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 1, no. 2, pp. 107–137, 1993.

[3] E. Mamdani, "Application of fuzzy algorithms for control of simple dynamic plant," *Proc. of the Institution of Electrical Engineers*, vol. 121, no. 12, pp. 1585–1588, 1974.

[4] E. Mamdani and S. Assilian, "An experiment in linguistic synthesis with a fuzzy logic controller," *Int. Journal of Man-Machine Studies*, vol. 7, pp. 1–13, 1975.

[5] M. Detyniecki, B. Moubêche, and B. Bouchon-Meunier, "On the paradoxical success of Mamdani's minimum-based inference," in *Combining Experimentation and Theory, A Hommage to Abe Mamdani*, ser. Studies in Fuzziness and Soft Computing, E. Trillas, P. Bonissone, L. Magdalena, and J. Kacprzyk, Eds. Springer, 2012, vol. 271, pp. 259–270.

[6] P.-N. Vo, M. Detyniecki, and B. Bouchon-Meunier, "Gradual Generalized Modus Ponens," in *Proc. of the IEEE Int. Conf. on Fuzzy Systems, FuzzIEEE13*, 2013.

[7] D. Dubois and H. Prade, "Gradual infrence rules in approximate reasoning," *Information sciences*, vol. 61, pp. 103–122, 1992.

[8] E. Van Broekhoven and B. De Baets, "Monotone Mamdani-Assilian models under mean of maxima defuzzification," *Fuzzy Sets and Systems*, vol. 159, no. 21, pp. 2819–2844, 2008.

[9] ——, "Only smooth rule bases can generate monotone Mamdani-Assilian models under center-of-gravity defuzzification," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 5, pp. 1157–1174, 2009.

[10] M. Stepnicka and B. De Baets, "Monotonicity of implicative fuzzy models," in *Proc. of the IEEE Int. Conf. on Fuzzy Systems*, 2010, pp. 1–7.

[11] K. M. Tay, C. Lim, and T. L. Jee, "Building monotonicity-preserving fuzzy inference models with optimization-based similarity reasoning and a monotonicity index," in *Proc. of the IEEE Int. Conf. on Fuzzy Systems*, 2012, pp. 1–8.

[12] B. Bouchon-Meunier and L. Valverde, "A fuzzy approach to analogical reasoning," *Soft Computing*, vol. 3, pp. 141–147, 1999.

[13] B. Bouchon-Meunier, R. Mesiar, C. Marsala, and M. Rifqi, "Compositional rule of inference as an analogical scheme," *Fuzzy Sets and Systems*, vol. 138, no. 1, pp. 53–68, 2003.

[14] G. Klir, "Measures of uncertainty and information," in *Fundamentals of fuzzy sets*, ser. The handbook of fuzzy sets, D. Dubois and H. Prade, Eds. Kluwer Academic Publishers, 2000, vol. 7, ch. 8, pp. 439–458.

[15] N. Pal and J. Bezdek, "Quantifying different facets of fuzzy uncertainty," in *Fundamentals of fuzzy sets*, ser. The handbook of fuzzy sets, D. Dubois and H. Prade, Eds. Kluwer Academic Publishers, 2000, vol. 7, ch. 9, pp. 459–479.

[16] B. Bouchon-Meunier, M. Rifqi, and S. Bothorel, "Towards general measures of comparison of objects," *Fuzzy Sets and Systems*, vol. 84, no. 2, pp. 143–153, 1996.