

# Weighted independence ratio of geometric distance graphs

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Thursday July 12, 2018

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Joint work with

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- 1 Context
  - Definitions
  - The Euclidean plane
  - Hadwiger-Nelson problem
- 2 Polytope norms in the plane
  - The problem
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- 3 Weighted graphs
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# Definitions

- Normed space  $E = (\mathbb{R}^n, \|\cdot\|)$ .
- A set  $A \in \mathbb{R}^n$  **avoids distance 1** iff  $\forall x, y \in A, \|x - y\| \neq 1$ .
- **(Upper) density** of a measurable set  $A$ :

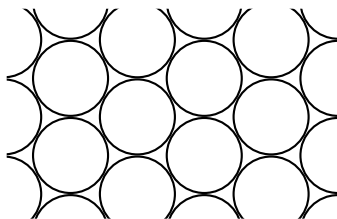
$$\delta = \limsup_{R \rightarrow \infty} \frac{\text{Vol}(A \cap [-R, R]^n)}{\text{Vol}([-R, R]^n)}.$$

- Maximum density of a set avoiding distance 1:

$$m_1(\mathbb{R}^n, \|\cdot\|) = \sup_{A \text{ avoiding } 1} \delta(A).$$

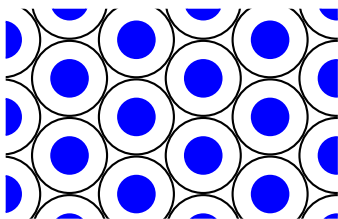
# Example

- Let  $\Lambda$  be a set of two pairwise disjoint balls of radius 1.



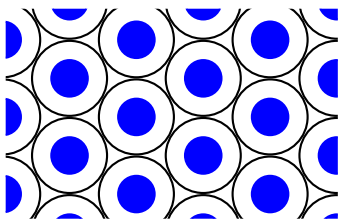
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- If the unit ball associated to a norm  $\| \cdot \|$  tiles  $\mathbb{R}^n$  ( $\| \cdot \|_\infty$  for example) :

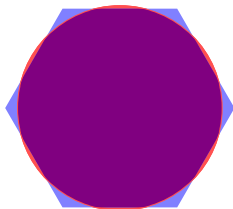
$$m_1(\mathbb{R}^n, \| \cdot \|) \geq \frac{1}{2^n}.$$

# Lower bounds

- The previous construction proves that  $m_1(\mathbb{R}^2, \|\cdot\|_2) \geq 0.9069/4 \geq 0.2267$ .

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- Croft (1967):  $m_1(\mathbb{R}^2, \|\cdot\|_2) \geq 0.229$ .





# Upper bounds

- Best upper bound :  $m_1(\mathbb{R}^2, \|\cdot\|_2) \leq 0.258795$  (Keleti, Matolcsi, de Oliveira Filho, Ruzsa, 2015).
- Erdős' conjecture :  $m_1(\mathbb{R}^2, \|\cdot\|_2) < 1/4$ .
- Generalization (Moser, Larman Rogers):  
 $m_1(\mathbb{R}^n, \|\cdot\|_2) < \frac{1}{2^n}$ .

# Definitions

## Chromatic number of a metric space

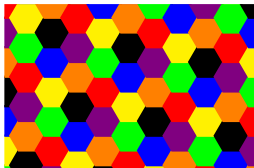
The **chromatic number**  $\chi$  of a metric space  $(X, d)$  is the smallest number of colours required to colour each point of  $X$  so that no two points at distance 1 share the same colour.

## Unit-distance graph

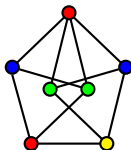
The **unit-distance graph** associated to a metric space  $(X, d)$  is the graph  $G$  such that  $V(G) = X$  and  $E(G) = \{\{x, y\} : d(x, y) = 1\}$ .

# The Euclidean plane

- $\chi(\mathbb{R}^2) \leq 7$ :



- $\chi(\mathbb{R}^2) \geq 4$  (Moser's spindle):



- De Grey (April 2018):  $\chi(\mathbb{R}^2) \geq 5$ .

# Measurable chromatic number

We define the **measurable chromatic number**  $\chi_m$  of a metric space  $(X, d)$  by adding the constraint that **the colour classes must be measurable set**.

$$\chi_m(\mathbb{R}^n, \|\cdot\|) \geq \frac{1}{m_1(\mathbb{R}^n, \|\cdot\|)}$$

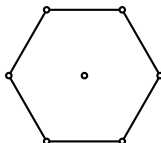
Euclidean plane:  $\chi_m(\mathbb{R}^2) \geq 5$ . (Falconer, 1981)  
Same bound as in the non-measurable case.

# Polytope norm

## Polytope norm

Let  $\mathcal{P}$  be a convex, symmetric polytope centered at 0 and of non-empty interior. The *polytope norm*  $\|\cdot\|_{\mathcal{P}}$  associated to  $\mathcal{P}$  is by definition

$$\|x\|_{\mathcal{P}} = \inf\{t \in \mathbb{R}^+ : x \in t\mathcal{P}\}.$$

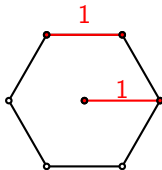


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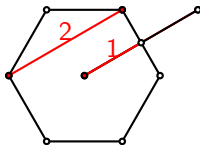


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If the unit ball associated to a norm  $\|\cdot\|_{\mathcal{P}}$  is a polytope that tiles  $\mathbb{R}^n$  (by translation),  $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \geq \frac{1}{2^n}$ .

### Conjecture (Bachoc, Robins)

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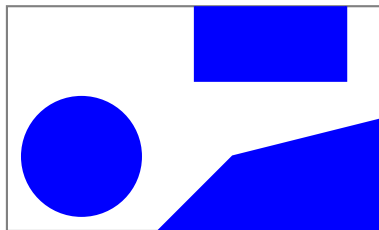
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### Theorem (Bachoc, Bellitto, Moustrou, Pêcher, 2017)

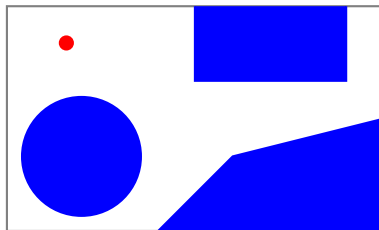
If  $\mathcal{P}$  tiles  $\mathbb{R}^2$  (by translation), then  $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) = \frac{1}{4}$ .

# Method



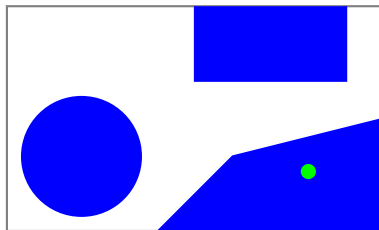
Set  $S$  of density  $\delta$ .  $X$  at random in  $\mathbb{R}^n$ :

# Method



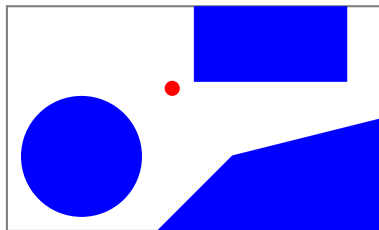
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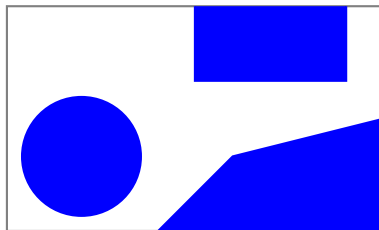
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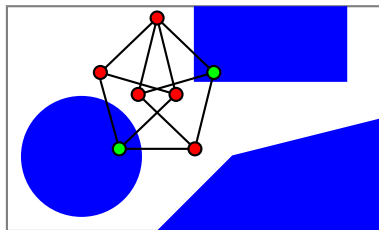
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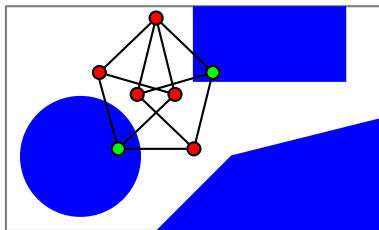
# Method



Set  $S$  of density  $\delta$ .  $X$  at random in  $\mathbb{R}^n$ :  $\mathbb{P}(X \in S) = \delta$ .

Unit-distance subgraph  $G$  at random in  $\mathbb{R}^n$ :

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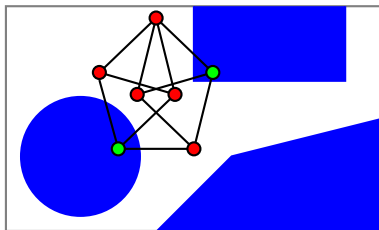
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$$\mathbb{E}(|V \cap S|) = |V| \times \delta.$$



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Unit-distance subgraph  $G$  at random in  $\mathbb{R}^n$ :

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If  $S$  avoids distance 1:  $|V \cap S| \leq \alpha(G) \rightarrow \delta \leq \frac{\alpha}{|V|}$ .

## Discretization lemma

For all unit-distance subgraph  $G$  in  $\mathbb{R}^n$ :

$$m_1(\mathbb{R}^n) \leq \overline{\alpha(G)} = \frac{\alpha(G)}{|V|}.$$

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## Determining $m_1(\mathbb{R}^2, \|\cdot\|_\infty)$



$K_4$  is a unit-distance subgraph.

$$m_1(\mathbb{R}^2, \|\cdot\|_\infty) = \frac{1}{4}.$$

# Definitions

**Weighting** of a graph:  $w : V \rightarrow \mathbb{R}^+$ .

**Weight** of a vertex set  $S$ :  $\sum_{v \in S} w(v)$ .

**Weighted independence number**  $\alpha_w(G)$  of a weighted graph  $G$ : maximum weight of an independent set.

**Weighted independence ratio**  $\overline{\alpha_w(G)} = \frac{\alpha_w(G)}{w(G)}$ .

# Definitions

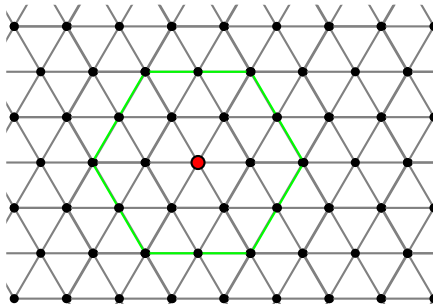
Optimal weighted independence ratio  $\alpha^*(G)$  of an **unweighted** graph  $G$ : minimum over all weightings of  $G$  of  $\overline{\alpha(G)}$ .

## Discretization lemma

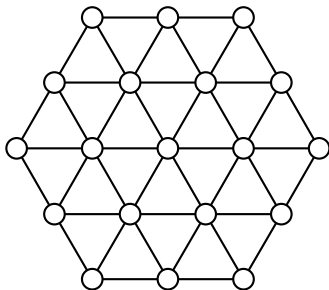
For all unit-distance subgraph  $G$  in  $\mathbb{R}^n$ :

$$m_1(\mathbb{R}^n) \leq \alpha^*(G).$$

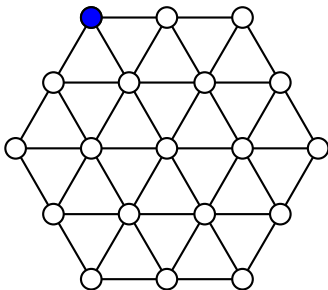
# The regular hexagon



# With an unweighted graph

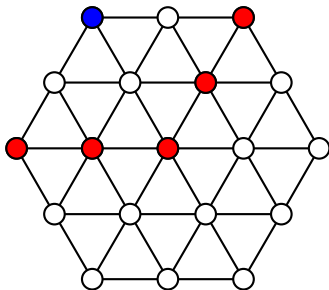


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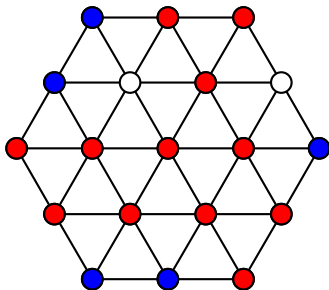




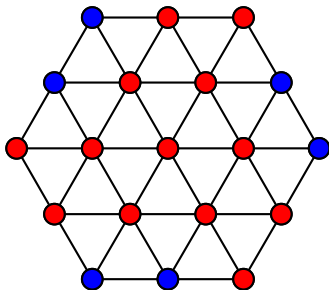
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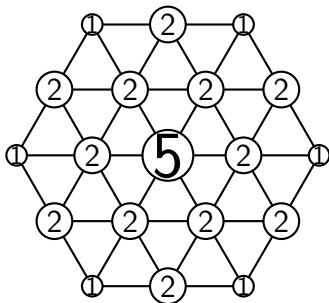


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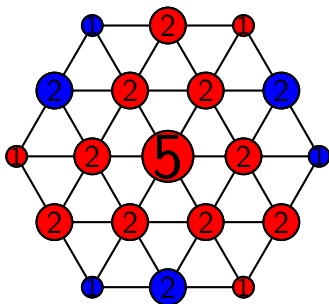


Provided bound :  $\frac{6}{19} \simeq 0.316$ .

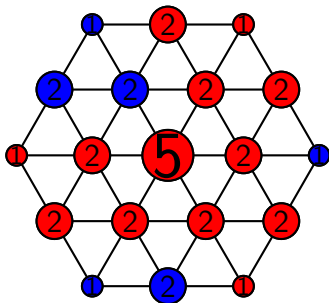
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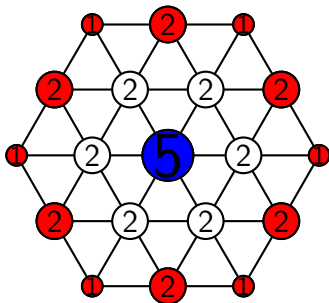
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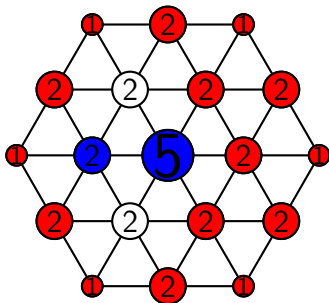
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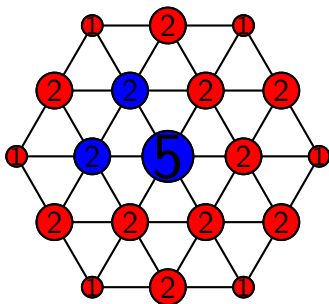


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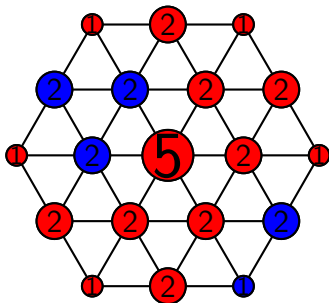




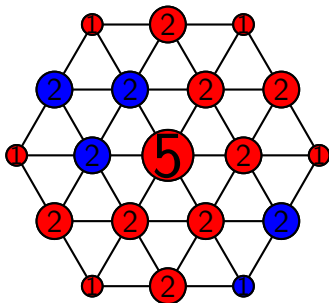
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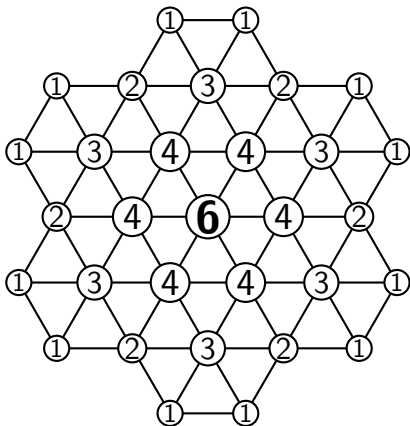


# With a weighted graph



Provided bound :  $\frac{9}{35} \simeq 0.257$ .

# Alternative proof of $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{H}}) \leq \frac{1}{4}$



This graph has weighted independence ratio  $\frac{1}{4}$ .

# Fractional colouring

## Chromatic number

The chromatic number  $\chi$  of a graph  $G$  is the smallest number  $a$  such that  $a$  colours are sufficient to colour each vertex of  $G$  in such a way that no two adjacent vertices share the same colour.

# Fractional colouring

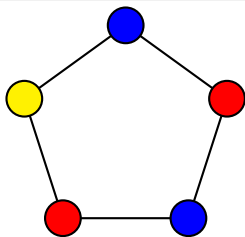
## Fractional chromatic number

The **fractional** chromatic number  $\chi_f$  of a graph  $G$  is the smallest number  $\frac{a}{b}$  such that  $a$  colours are sufficient to **assign**  $b$  colours to each vertex of  $G$  in such a way that no two adjacent vertices share a common colour.

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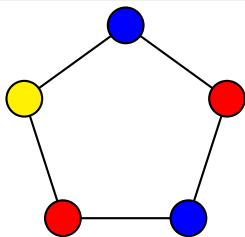


$$\chi(C_5) = 3$$

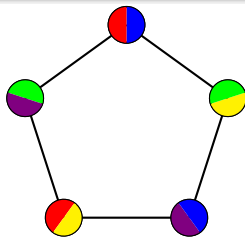
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$$\chi(C_5) = 3$$



$$\chi_f(C_5) = \frac{5}{2}$$



# Relation between these parameters

$$\alpha^*(G) = \frac{1}{\chi_f(G)}$$

$$\frac{1}{\chi_m(\mathbb{R}^n, \|\cdot\|)} \leq m_1(\mathbb{R}^n, \|\cdot\|) \leq \frac{1}{\chi_f(\mathbb{R}^n, \|\cdot\|)}.$$

# Algorithms

- Design of an efficient algorithm for the fractional chromatic number, based on linear programming, especially optimized for our geometric instances.
- Method to build graphs of high fractional chromatic number and of size within the limit of our computational power.

# The Euclidean plane

Cranston, Rabern (2017):  $\chi_f(\mathbb{R}^2) \geq \frac{76}{21} \geq 3.61904$ .  
 $\Rightarrow m_1(\mathbb{R}^2) \leq 0.276316$ .

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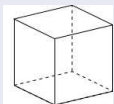
## Theorem (Bellitto, Pêcher, Sedillot)

$$\chi_f(\mathbb{R}^2) \geq 3.8977.$$

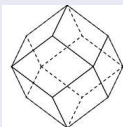
$$m_1(\mathbb{R}^2) \leq 0.25656.$$

# Regular 3-dimensional parallelotetra

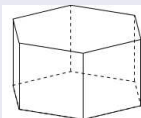
Current results (Bachoc, Bellitto, Moustrou, Pêcher)



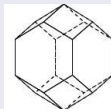
$$m_1 = \frac{1}{8}$$



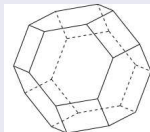
$$m_1 = \frac{1}{8}$$



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$$m_1 \leq 0.130443$$

Thank you!