Optimal weighting to minimize the independence ratio of a graph

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Joint work with
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Definitions

- Normed space $E = (\mathbb{R}^n, \|\cdot\|)$.
- A set $A \in \mathbb{R}^n$ avoids distance 1 iff $\forall x, y \in A, \|x - y\| \neq 1$.
- (Upper) density of a measurable set A:

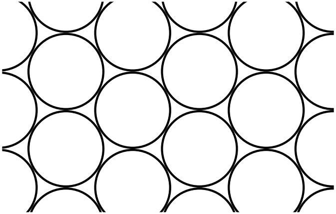
$$\delta = \limsup_{R\to\infty} \frac{\mathrm{Vol}(A \cap [-R, R]^n)}{\mathrm{Vol}([-R, R]^n)}.$$

- Maximum density of a set avoiding distance 1:

$$m_1(\mathbb{R}^n, \|\cdot\|) = \sup_{A \text{ avoiding } 1} \delta(A).$$

Example

- Let Λ be a set of two pairwise disjoint balls of radius 1.



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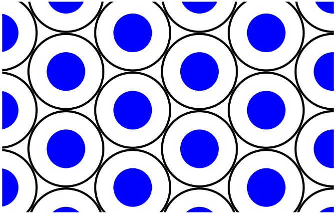
- We build from Λ a set avoiding distance 1 of density $\frac{\delta(\Lambda)}{2^n}$.



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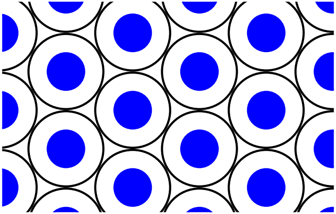

- If the unit ball associated to a norm $\|\cdot\|$ tiles $\mathbb{R}^n$ ($\|\cdot\|_\infty$ for example) :

$$m_1(\mathbb{R}^n, \|\cdot\|) \geqslant \frac{1}{2^n}.$$

Lower bounds

- The previous construction proves that $m_1(\mathbb{R}^2, \|\cdot\|_2) \geqslant 0.9069/4 \geqslant 0.2267$.

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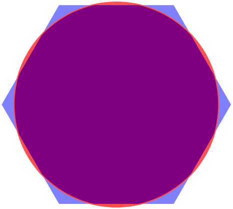
- Croft (1967) $m_1(\mathbb{R}^2, \|\cdot\|_2) \geqslant 0.229$.



Upper bounds

- Best upper bound : $m_1(\mathbb{R}^2, \|\cdot\|_2) \leqslant 0.258795$ (Keleti, Matolcsi, de Oliveira Filho, Ruzsa, 2015).
- Erdös' conjecture : $m_1(\mathbb{R}^2, \|\cdot\|_2) < 1/4$.
- Generalization (Moser, Larman Rogers): $m_1(\mathbb{R}^n, \|\cdot\|_2) < \frac{1}{2^n}$.

Definitions

Chromatic number of a metric space

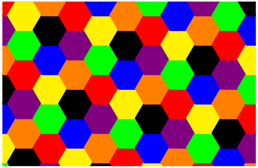
The chromatic number χ of a metric space(X, d) is the smallest number of colours required to colour each point of X so that no two points at distance 1 share the same colour.

Unit-distance graph

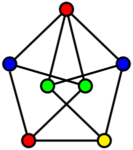
The unit-distance graph associated to a metric space (X, d) is the graph G such that $V(G) = X$ and $E(G) = \{\{x, y\} : d(x, y) = 1\}$.

The Euclidean plane

- $\chi(\mathbb{R}^2) \leqslant 7$:



- $\chi(\mathbb{R}^2) \geqslant 4$ (Moser's spindle):



- De Grey (April 2018): $\chi(\mathbb{R}^2) \geqslant 5$.

Measurable chromatic number

We define the measurable chromatic number χ_m of a metric space (X, d) by adding the constraint that the colour classes must be measurable set.

$$\chi_m(\mathbb{R}^n, \|\cdot\|) \geqslant \frac{1}{m_1(\mathbb{R}^n, \|\cdot\|)}$$

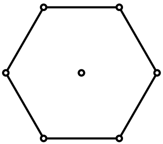
Euclidean plane: $\chi_m(\mathbb{R}^2) \geqslant 5$. (Falconer, 1981)
Same bound as in the non-measurable case.

Polytope norm

Polytope norm

Let $\mathcal{P}$ be a convex, symmetric polytope centered at 0 and of non-empty interior. The *polytope norm* $\|\cdot\|_{\mathcal{P}}$ *associated to* $\mathcal{P}$ is by definition

$$\|x\|_{\mathcal{P}} = \inf\{t \in \mathbb{R}^+ : x \in t\mathcal{P}\}.$$



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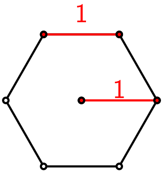

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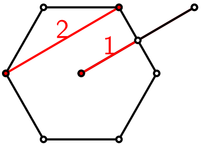

If the unit ball associated to a norm $\|\cdot\|_{\mathcal{P}}$ is a polytope that tiles $\mathbb{R}^n$ (by translation), $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \geqslant \frac{1}{2^n}$.

Conjecture (Bachoc, Robins)

If $\mathcal{P}$ tiles $\mathbb{R}^n$ (by translation), then $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) = \frac{1}{2^n}$.

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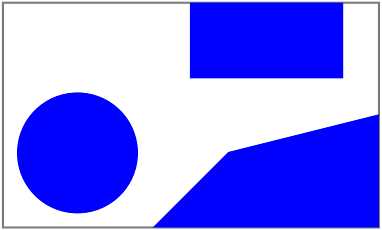
Theorem (Bachoc, Bellitto, Moustrou, Pêcher, 2017)

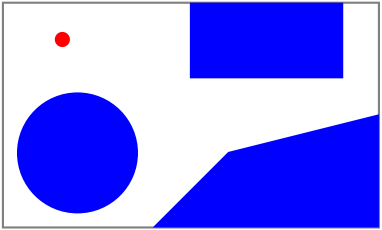
If $\mathcal{P}$ tiles $\mathbb{R}^2$ (by translation), then $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) = \frac{1}{4}$.

Method



Set S of density δ. X at random in $\mathbb{R}^n$:

Method



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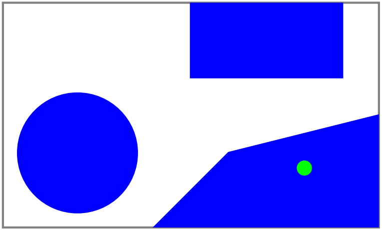

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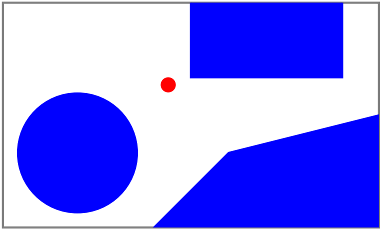

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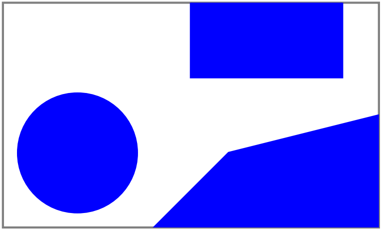
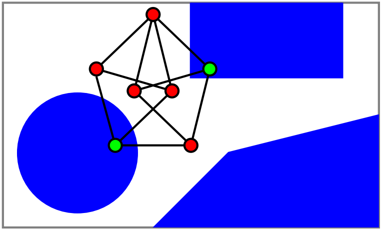


Set S of density δ. X at random in $\mathbb{R}^n$: $\mathbb{P}(X \in S) = \delta$.

Method



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Unit-distance subgraph G at random in $\mathbb{R}^n$:

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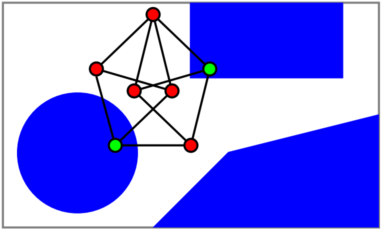

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$\mathbb{E}(|V \cap S|) = |V| \times \delta$.

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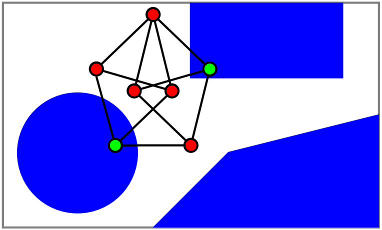

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Unit-distance subgraph G at random in $\mathbb{R}^n$:
$\mathbb{E}(|V \cap S|) = |V| \times \delta$.

If S avoids distance 1: $|V \cap S| \leqslant \alpha(G) \rightarrow \delta \leqslant \frac{\alpha}{|V|}$.

Discretization lemma

For all unit-distance subgraph G in $\mathbb{R}^n$:

$$m_1(\mathbb{R}^n) \leqslant \overline{\alpha(G)} = \frac{\alpha(G)}{|V|}.$$

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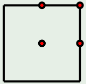
Determining $m_1(\mathbb{R}^2, \|\cdot\|_\infty)$



K_4 is a unit-distance subgraph.

$$m_1(\mathbb{R}^2, \|\cdot\|_\infty) = \frac{1}{4}.$$

Definitions

Weighting of a graph: $w : V \to \mathbb{R}^+$.

Weight of a vertex set S: $\sum_{v \in S} w(v)$.

Weighted independence number $\alpha_w(G)$ of a weighted graph G: maximum weight of an independent set.

Weighted independence ratio $\overline{\alpha_w(G)} = \frac{\alpha_w(G)}{w(G)}$.

Definitions

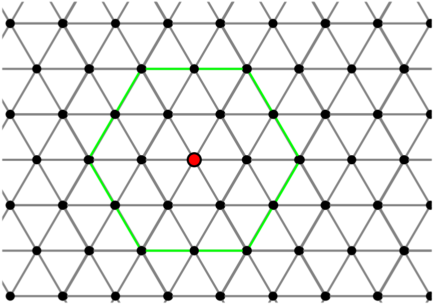
Optimal weighted independence ratio $\alpha^*(G)$ of an unweighted graph G: minimum over all weightings of G of $\overline{\alpha(G)}$.

Discretization lemma

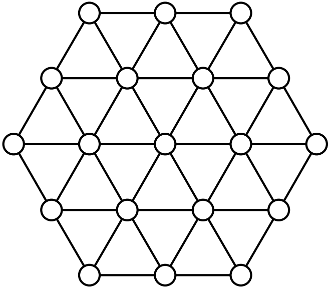
For all unit-distance subgraph G in $\mathbb{R}^n$:

$$m_1(\mathbb{R}^n) \leqslant \alpha^*(G).$$

The regular hexagon



With an unweighted graph



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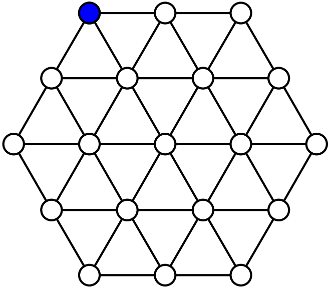

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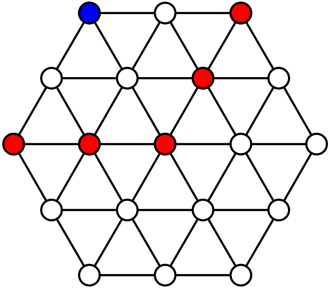

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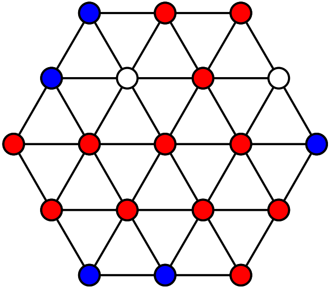

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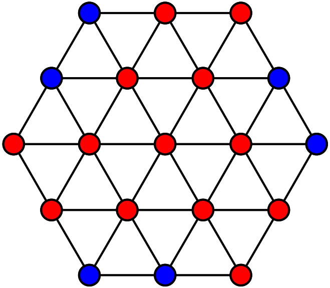
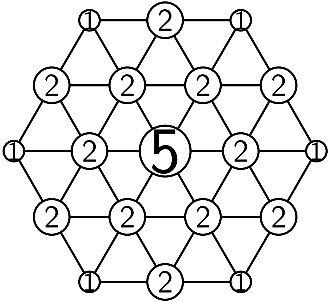


Provided bound : $\frac{6}{19} \simeq 0.316$.

With a weighted graph



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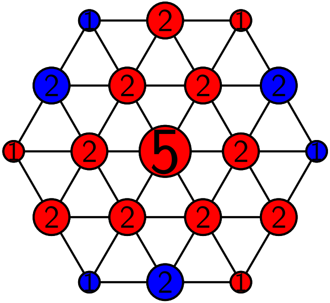

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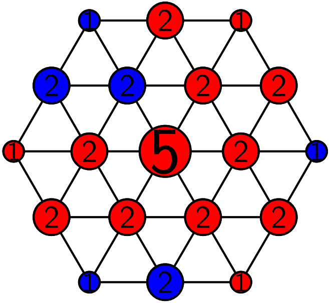

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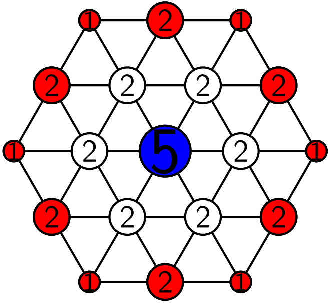

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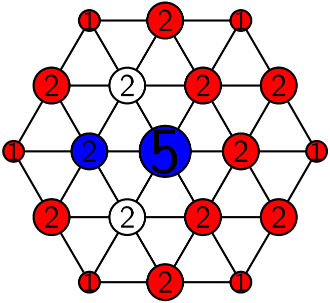

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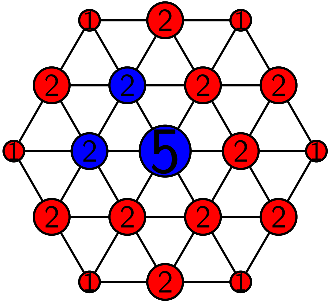

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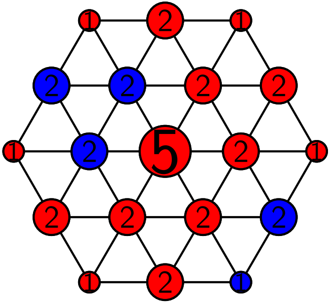

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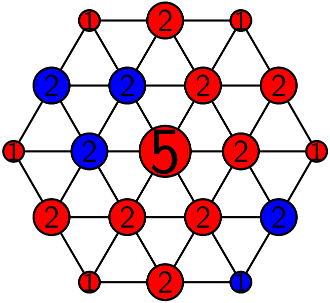
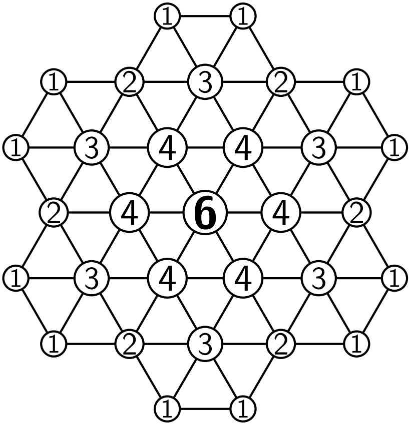


Provided bound : $\frac{9}{35} \simeq 0.257$.

Alternative proof of $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{H}}) \leqslant \frac{1}{4}$



This graph has weighted independence ratio $\frac{1}{4}$.

Fractional colouring

Chromatic number

The chromatic number χ of a graph G is the smallest number a such that a colours are sufficient to colour each vertex of G in such a way that no two adjacent vertices share the same colour.

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The fractional chromatic number χ_f of a graph G is the smallest number $\frac{a}{b}$ such that a colours are sufficient to assign b colours to each vertex of G in such a way that no two adjacent vertices share a common colour.

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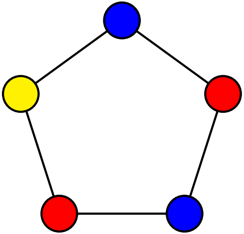

$\chi(C_5) = 3$

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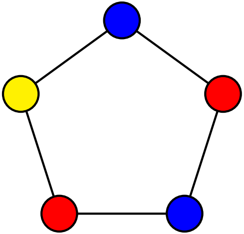

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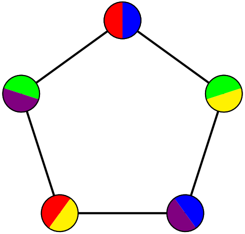

$\chi_f(C_5) = \frac{5}{2}$

Fractional clique number

Fractional clique (Godsil and Royle, 2001)

A fractional clique is a weight distribution on the vertices of a graph such that no independent set has weight more than 1.
The weight of a fractional clique is the total weight of the graph under the weighting defined by the clique.
The fractional clique number ω_f of a graph is the maximum weight of a fractional clique.

Relation between these parameters

By strong duality, $\chi_f(G) = \omega_f(G)$.

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$$\frac{1}{\chi_m(\mathbb{R}^n, \|\cdot\|)} \leqslant m_1(\mathbb{R}^n, \|\cdot\|) \leqslant \frac{1}{\chi_f(\mathbb{R}^n, \|\cdot\|)}.$$

LP formulation of χ and χ_f

$\mathcal{S}$: set of all independent sets in the graph.
For all $I \in S$, x_I indicates whether I is a colour class.

$$\begin{cases} \text{minimize } \sum_{I \in \mathscr{S}} x_I \text{ subject to} \\ \forall v \in V, \sum_{I \in \mathscr{S} : x \in I} x_I = 1 \end{cases}$$

x_I binary $\rightarrow$ chromatic number.
x_I real $\rightarrow$ fractional chromatic number.

LP reformulation of α^*

For every vertex v, w_v indicates the weight of v:

$$\begin{cases} \text{minimize } M \\ \sum_{v \in V} w_v = 1 \\ \forall I \in \mathcal{S}, \sum_{v \in I} w_v \leqslant M \end{cases}$$

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But generating $\mathcal{S}$ can be really long!

Outline of the algorithm

We start with $\mathcal{S} = \varnothing$ and a uniform weight distribution W on the vertices of the graph.

- We add to $\mathcal{S}$ a maximum weight independent set for W (gives an upper bound on α^*).
- We define W as the weight distribution that minimizes the maximum weight of sets of $\mathcal{S}$ (gives a lower bound on α^*).

We stop when the two bounds coincide.

Optimization

- Giving the same weight to all the vertices of the same orbit.
- Generating quickly interesting sets in $\mathcal{S}$. With parallelohedron norms, we can use the periodicity of solutions.
- Constraints based on the maximal cliques of the graph or subgraphs of special interest (in the Euclidean case, Moser's spindle).

The Euclidean plane

Cranston, Rabern (2017): $\chi_f(\mathbb{R}^2) \geqslant \frac{76}{21} \geqslant 3.61904$.
$\Rightarrow m_1(\mathbb{R}^2) \leqslant 0.276316$.

The Euclidean plane

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Exoo, Ismailescu (unpublished): $\chi_f(\mathbb{R}^2) \geqslant \frac{383}{102} \geqslant 3.75491$.
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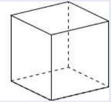
Theorem (Bellitto, Pêcher, Sedillot)

$$\chi_f(\mathbb{R}^2) \geqslant 3.89366.$$

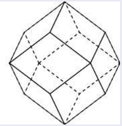
$$m_1(\mathbb{R}^2) \leqslant 0.256828.$$

Regular 3-dimensional parallelohedra

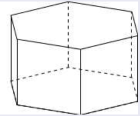
Current results (Bachoc, Bellitto, Moustrou, Pêcher)



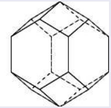
$m_1 = \frac{1}{8}$



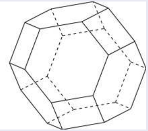
$m_1 = \frac{1}{8}$



$m_1 = \frac{1}{8}$



$m_1 = \frac{1}{8}$



$m_1 \leqslant 0.130443$

Thank you!