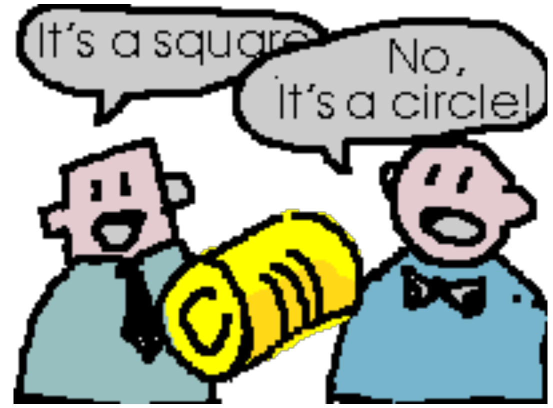

A survey on discrete tomography

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1/5/2014 München

Applications of Tomography

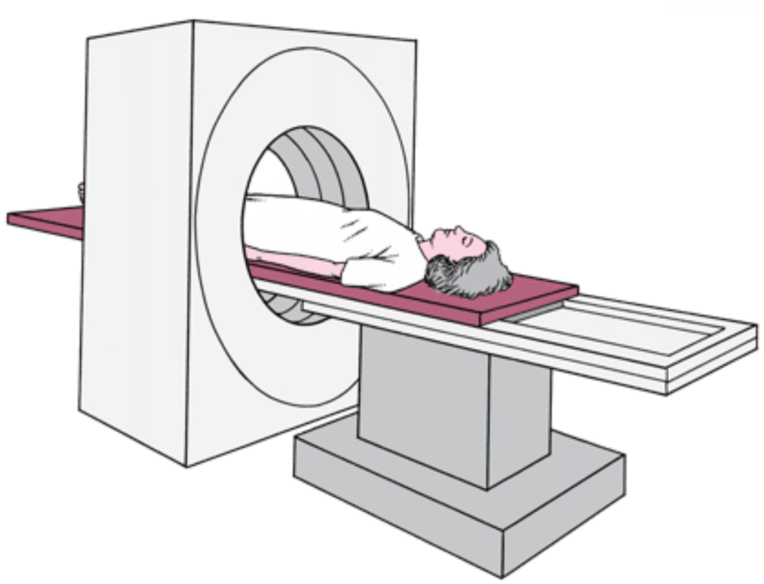
- given: 3 dimensional object
- input: several 2 dimensional projections
- output: 3D model of the object

discrete tomography

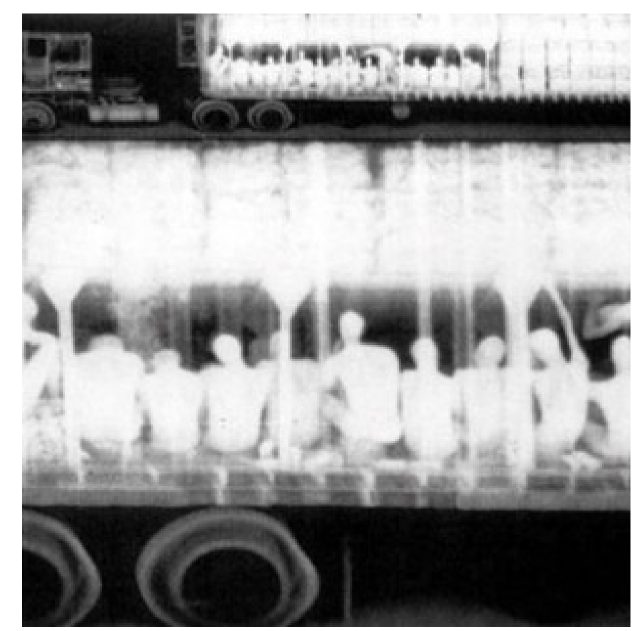
few projections
object has discrete nature (pixelized)

Larry Shepp 1994

non destructive quality control in manufacturing



medical application



smuggle detection

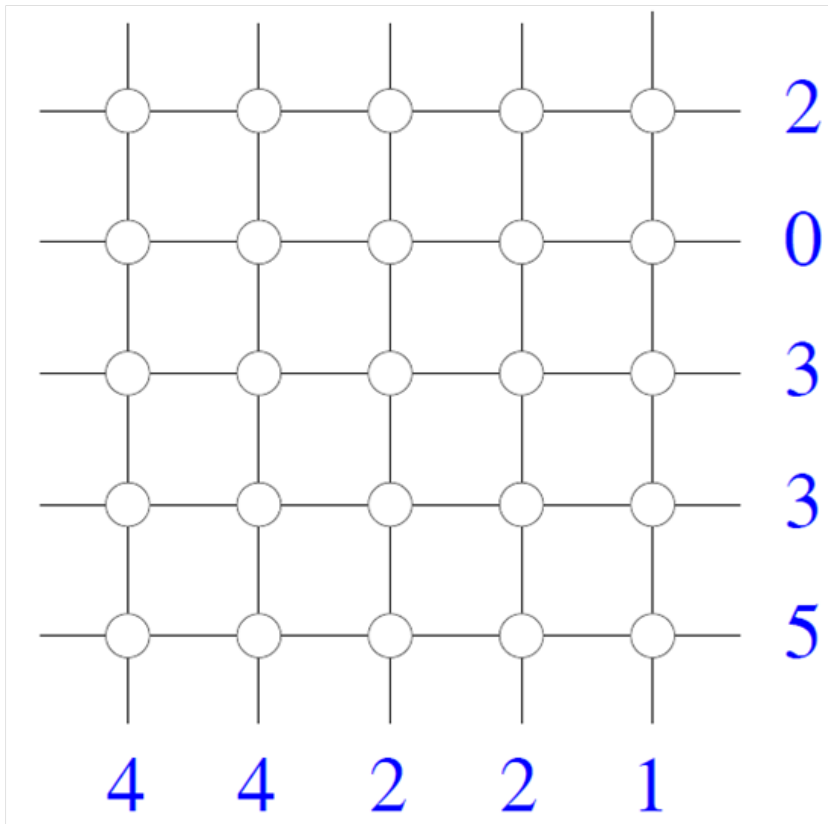
Figures: The Merck Manuals; M. Simon and C. Sauerwein; Erica Seccombe

Outline

- Ryser's theorem
- a reduction to 2SAT
- reconstructing domino tilings

A basic problem

input



output

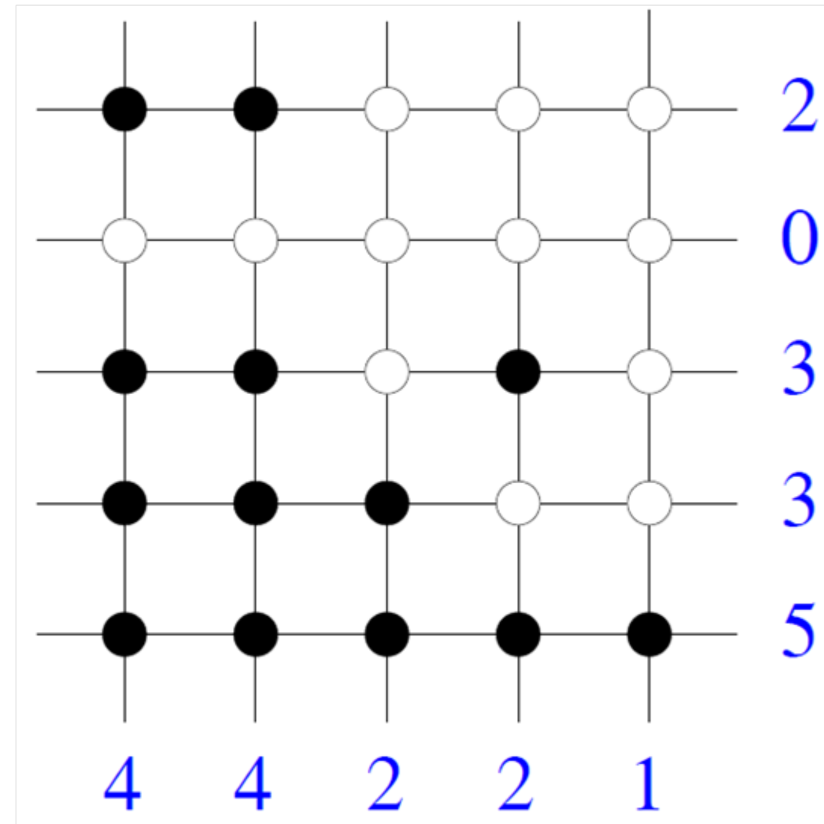
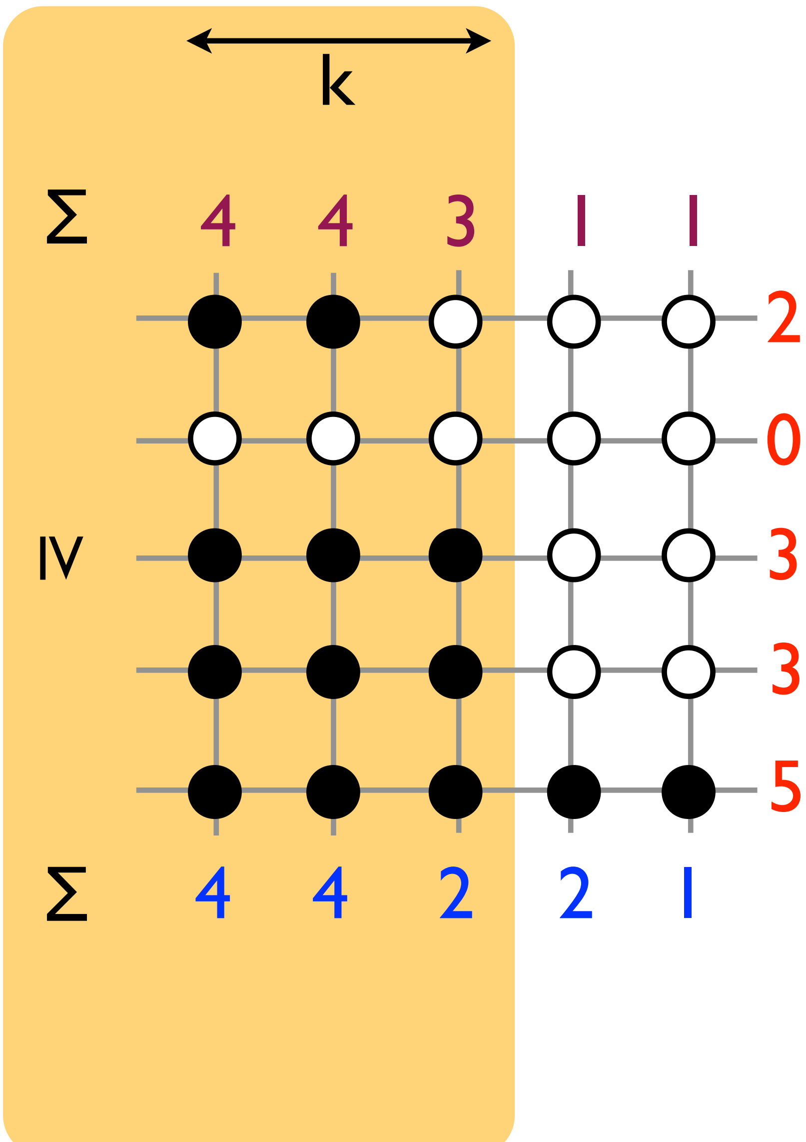


Figure: Andreas Alpers on Wikipedia

can be solved in polynomial time [Ryser'57]

- for every row (in any order), place the required number of pebbles always in a column with maximal remaining requirement
- is a transportation problem in complete bi-partite graph with unit capacities
- is a bi-partite matching problem
- has a solution iff *dual of column projections dominate row projections*

Ryser's theorem



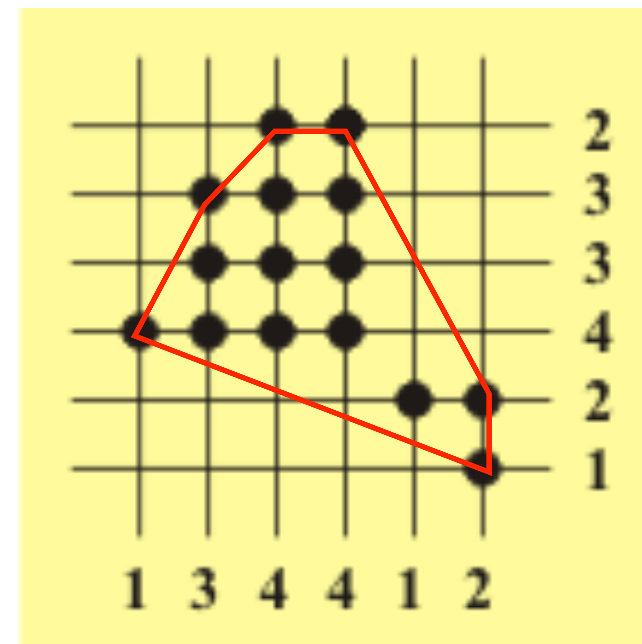
- [Thm] there is a pebble placement with row projection r and column projection s
  iff
  the dual of r dominates s
- [Def] The dual of r is the column projection of the leftmost placement with row projection r
- [Def] The dual r* dominates s if for every k, the prefix sums over i=1…k satisfy $\Sigma r^*_i \geq \Sigma s_i$
- [proof only if] r* dominates the column proj of any placement with row proj. r
- [proof if] right shifting a pebble in the first column k with $\Sigma_{i=1}^{k} r^*_i > \Sigma_{i=1}^{k} s_i$ preserves dominance and makes vectors closer

adding convexity constraints

- In general solution is not unique
  think: solutions to (1,...,1),(1,...,1) are all permutation matrices
- So it makes sense to impose properties on solutions:
- connected: pebbles should form a connected set (4-neighborhood)
- convex: convex hull of pebbles should contain no empty cell
- weaker hv-convex: every row or column should contain a single conting

uous sequence of pebbles

example



not connected
not convex
but hv-convex

Figure: Attila Kuba, Péter Balázs, Antal Nagy, Emese Balogh

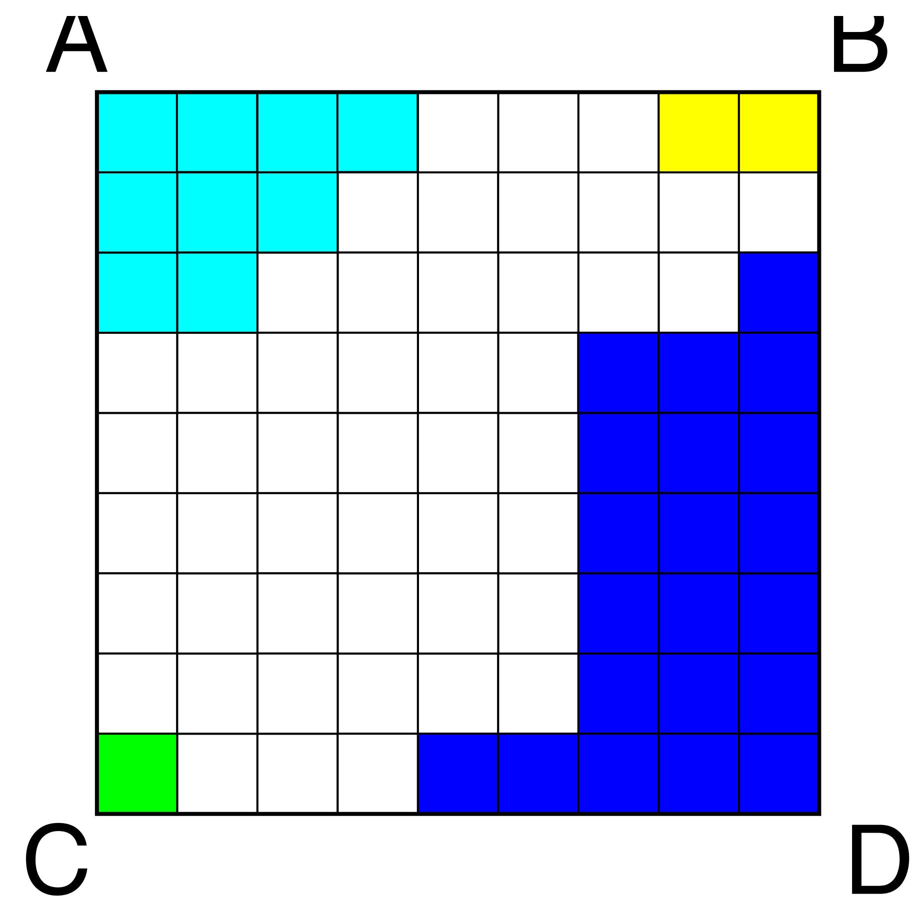
complexity status

	unconstrained	connected
unconstrained	polynomial [Ryser'57]	NP-complete [Woeginger'96]
hv-convex	NP-complete [Barcucci,delLungo,Nivat,Pinzani'96]	polynomial [Barcucci,delLungo,Nivat,Pinzani'96]
convex		polynomial [Brunetti,Daurat,Kuba'2006..08] (needs proj. from 4 dir)

An algorithm for reconstructing hv-convex connected sets

[Chrobak,D'99] exploiting tools from [Barcucci,delLungo,Nivat,Pinzani'96]

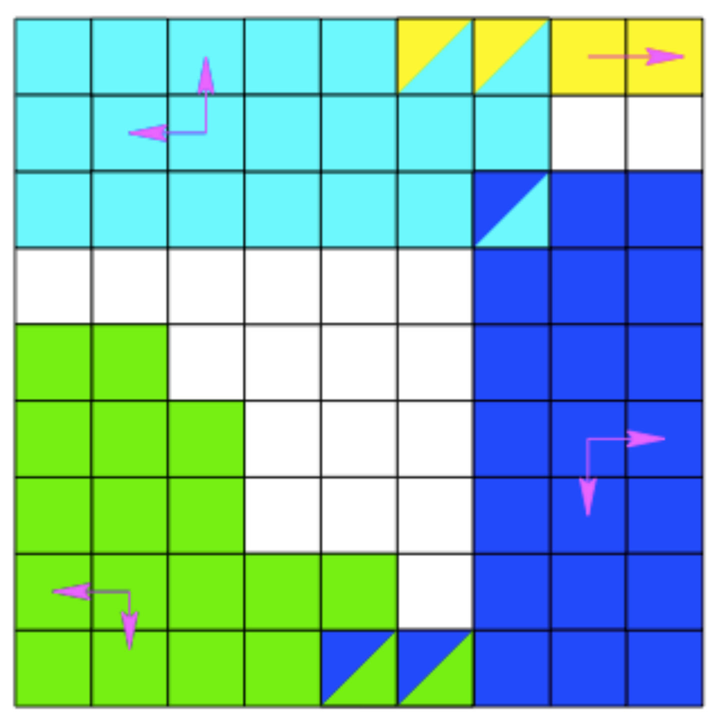
- Reduce to **2-SAT** as follows
- Wlog assume all projections are positive
- Divide cells not in solution into four regions
- Define indicator boolean variables A_{ij},B_{ij},C_{ij},D_{ij} for each cell i,j



- impose following constraints …

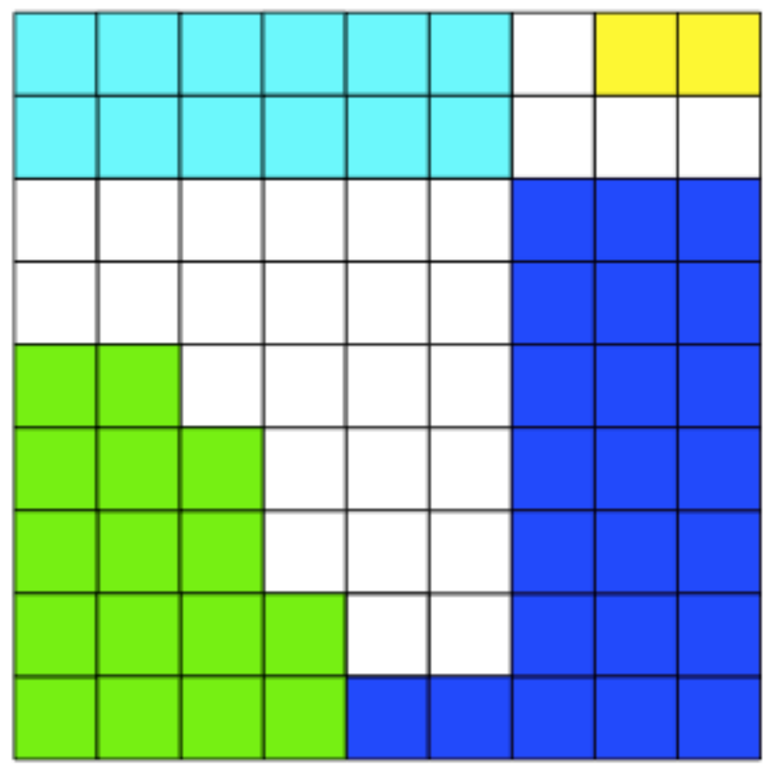
Impose set defined by A (B,C,D) corresponds to a corner region

$$Cor = \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow A_{i-1,j} & A_{i,j} \Rightarrow A_{i,j-1} \\ B_{i,j} \Rightarrow B_{i-1,j} & B_{i,j} \Rightarrow B_{i,j+1} \\ C_{i,j} \Rightarrow C_{i+1,j} & C_{i,j} \Rightarrow C_{i,j-1} \\ D_{i,j} \Rightarrow D_{i+1,j} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$



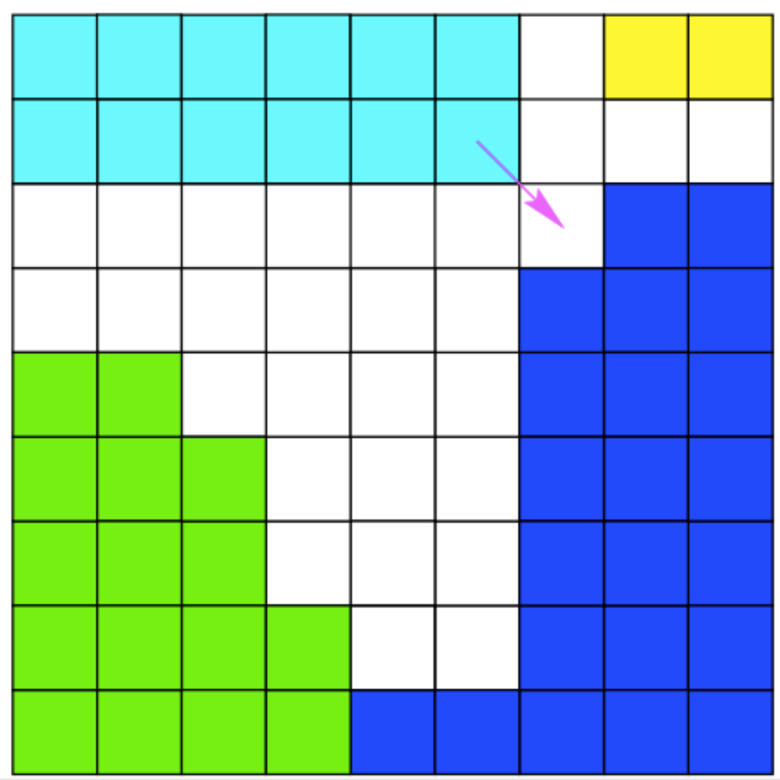
Make sure corners are disjoint

$$Dis = \bigwedge_{i,j} \{X_{i,j} \Rightarrow \overline{Y}_{i,j} : \text{ for distinct symbols } X, Y \in \{A, B, C, D\}\}$$



Make sure cells left by corners are connected

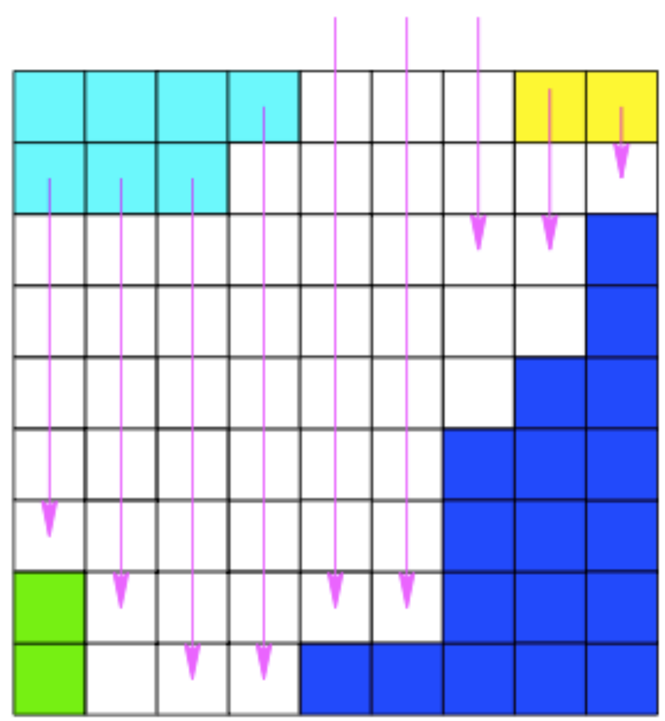
$$Con = \bigwedge_{i,j} \{A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \overline{C}_{i+1,j-1}\}$$



require column projections to be at least the given numbers

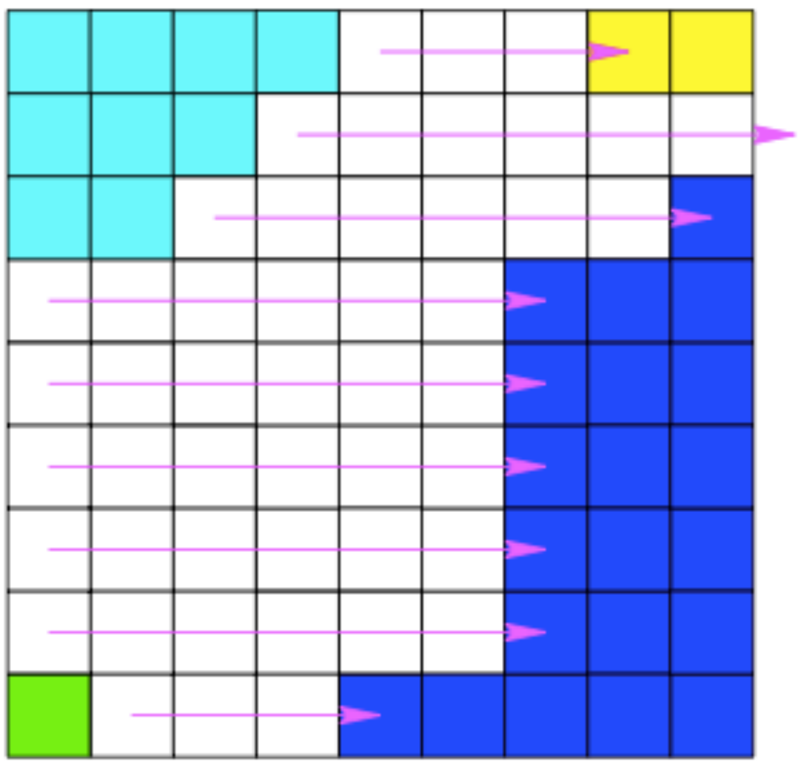
$$LBC = \bigwedge_{i,j} \{(A_{i,j} \vee B_{i,j}) \Rightarrow (\overline{C}_{i+c_j,j} \wedge \overline{D}_{i+c_j,j})\} \wedge \bigwedge_{j} \{\overline{C}_{c_j,j} \quad \overline{D}_{c_j,j}\}$$

$$= \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \overline{C}_{i+c_j,j} & A_{i,j} \Rightarrow \overline{D}_{i+c_j,j} \\ B_{i,j} \Rightarrow \overline{C}_{i+c_j,j} & B_{i,j} \Rightarrow \overline{D}_{i+c_j,j} \end{array} \right\} \wedge \bigwedge_{j} \{\overline{C}_{c_j,j} \quad \overline{D}_{c_j,j}$$



require row projections to be at most the given numbers

$$UBR = \bigwedge_{i,j} \{(\overline{A}_{i,j} \wedge \overline{C}_{i,j}) \Rightarrow (B_{i,j+r_i} \vee D_{i,j+r_i})\}$$

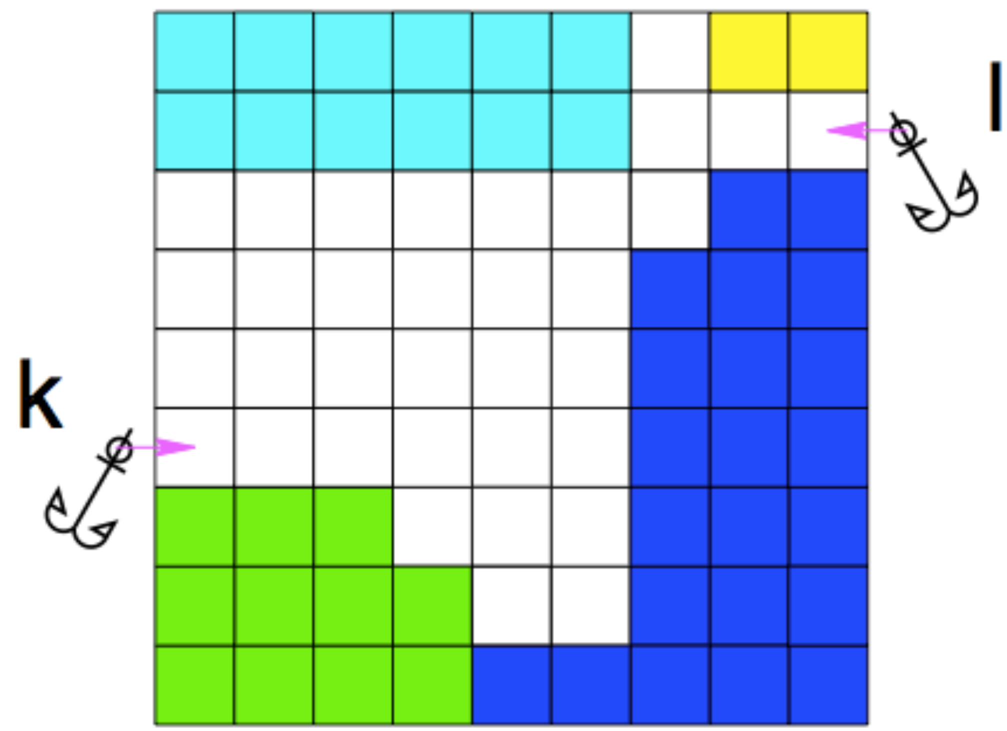


Problem: does not transform into 2-SAT clauses

work-around: fix cells on border to be part of solution

OK: only $O(n^2)$ possible *anchor* positions to test

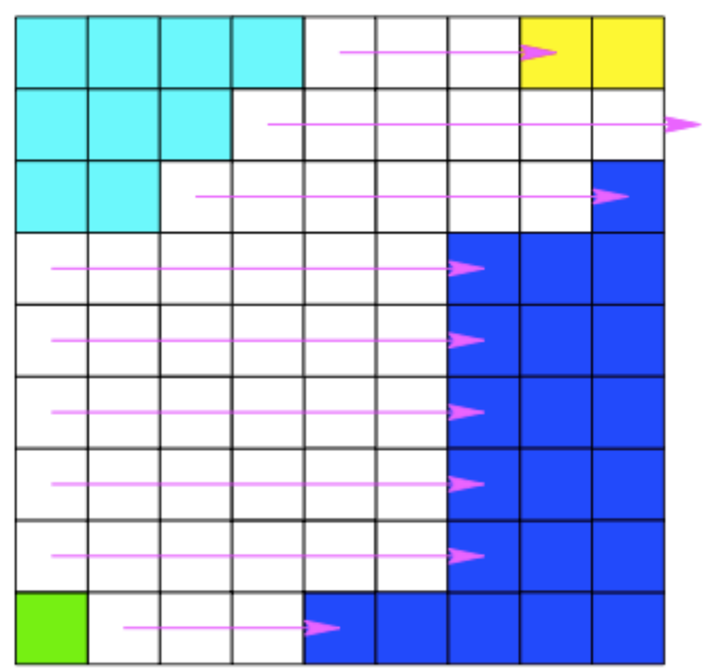
$$Anc = (\overline{A}_{k,1} \wedge \overline{B}_{k,1} \wedge \overline{C}_{k,1} \wedge \overline{D}_{k,1}) \wedge (\overline{A}_{l,n} \wedge \overline{B}_{l,n} \wedge \overline{C}_{l,n} \wedge \overline{D}_{l,n})$$



now with the anchors, splits nicely into 2-SAT clauses

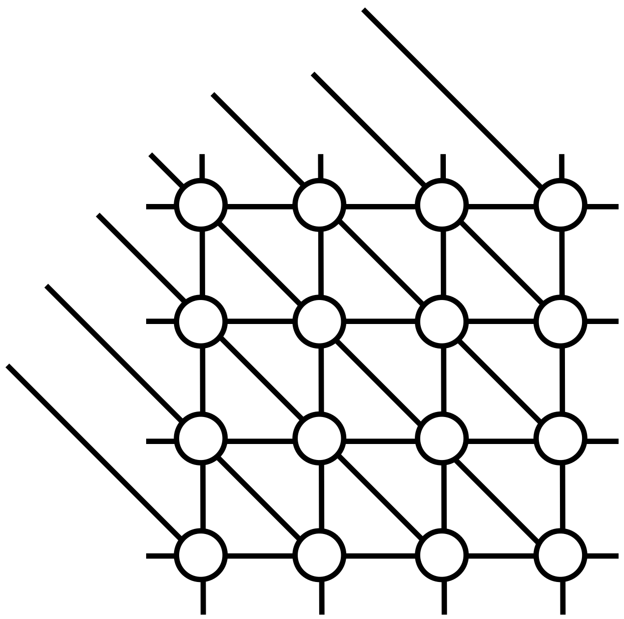
$$UBR = \bigwedge_{i,j} \{(\overline{A}_{i,j} \wedge \overline{C}_{i,j}) \Rightarrow (B_{i,j+r_i} \vee D_{i,j+r_i})\}$$

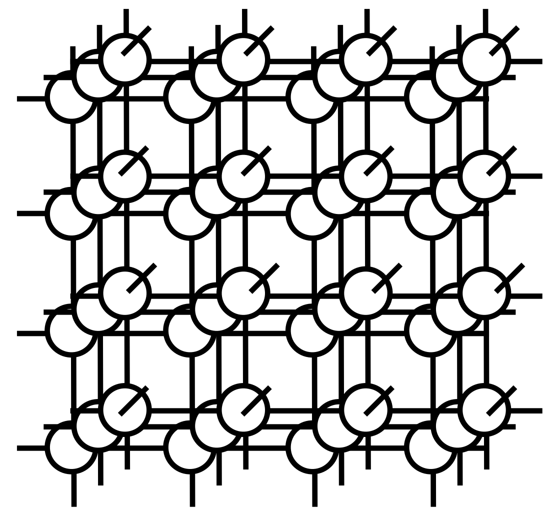
$$= \bigwedge_{j} \begin{cases} \bigwedge_{i \leq \min\{k,l\}} \overline{A}_{i,j} \Rightarrow B_{i,j+r_i} & \bigwedge_{k \leq i \leq l} \overline{C}_{i,j} \Rightarrow B_{i,j+r_i} \\ \bigwedge_{l \leq i \leq k} \overline{A}_{i,j} \Rightarrow D_{i,j+r_i} & \bigwedge_{\max\{k,l\} \leq i} \overline{C}_{i,j} \Rightarrow D_{i,j+r_i} \end{cases}$$

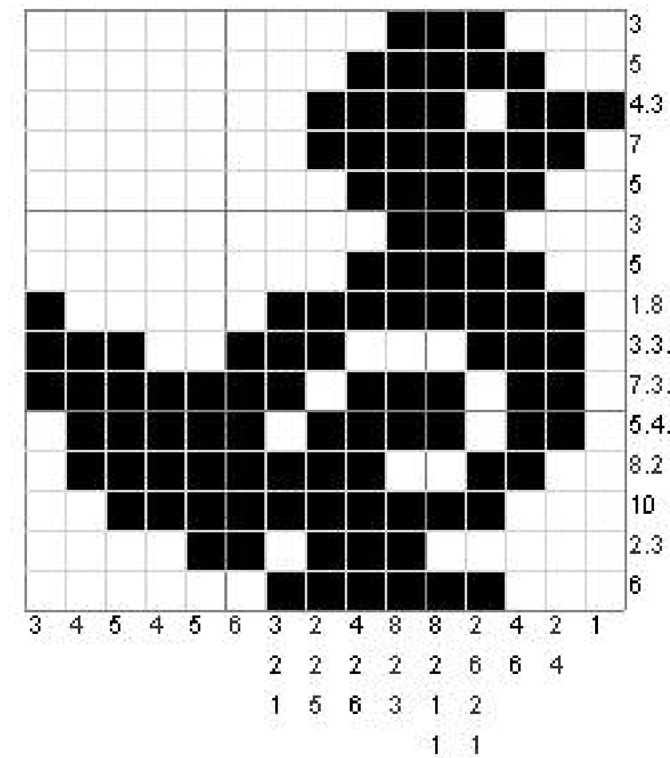


Final Complexity (quadratic in grid size):
$O(n^2)$ 2-SAT formulaes to check, each over $O(n^2)$ variables

many variants are NP-hard

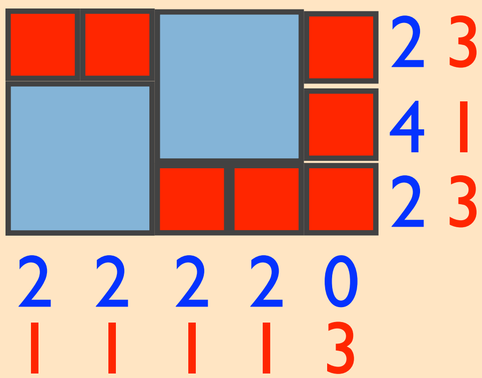






- more than 2 projections
- 3-dimensions (colors)
- nonograms

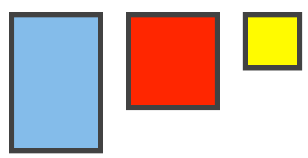
Tomography of tilings



- fix a set S of tile types
- now the projections indicate for every row/column how many grid cells are covered by a type of tile
- complexity of the reconstruction problem depends on S

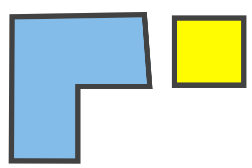
set S of tile types | complexity status

NP-hard

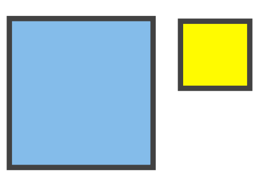


- |S|≥3 plus some cond.

[Gardner,Gritzmann,Prangenberg'98]
[D,Guiñez,Matamala'09]
[Chrobak,D,Guiñez,Lozano,Nguyen'10]

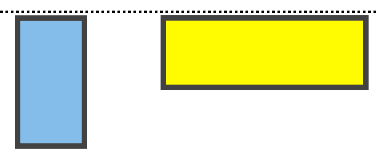


- S consists of a cell and a tile different from a bar



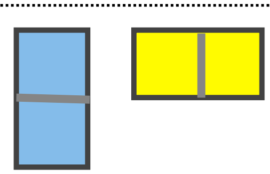
- S consists of 2 rectangles, and one has width,height ≥2

Open



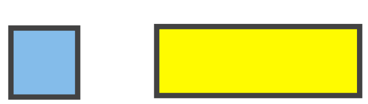
- S consits of a 1xa tile and a bx1 tile with a>b≥2

polynomial



- S consists of 2 dominos

[Thiant'06]



- S consists of a cell and a 1xa tile with arbitrary a

[Picouleau'01]
[D,Goles,Rapaport,Rémila'03]

a polynomial algorithm for reconstruction of domino tilings

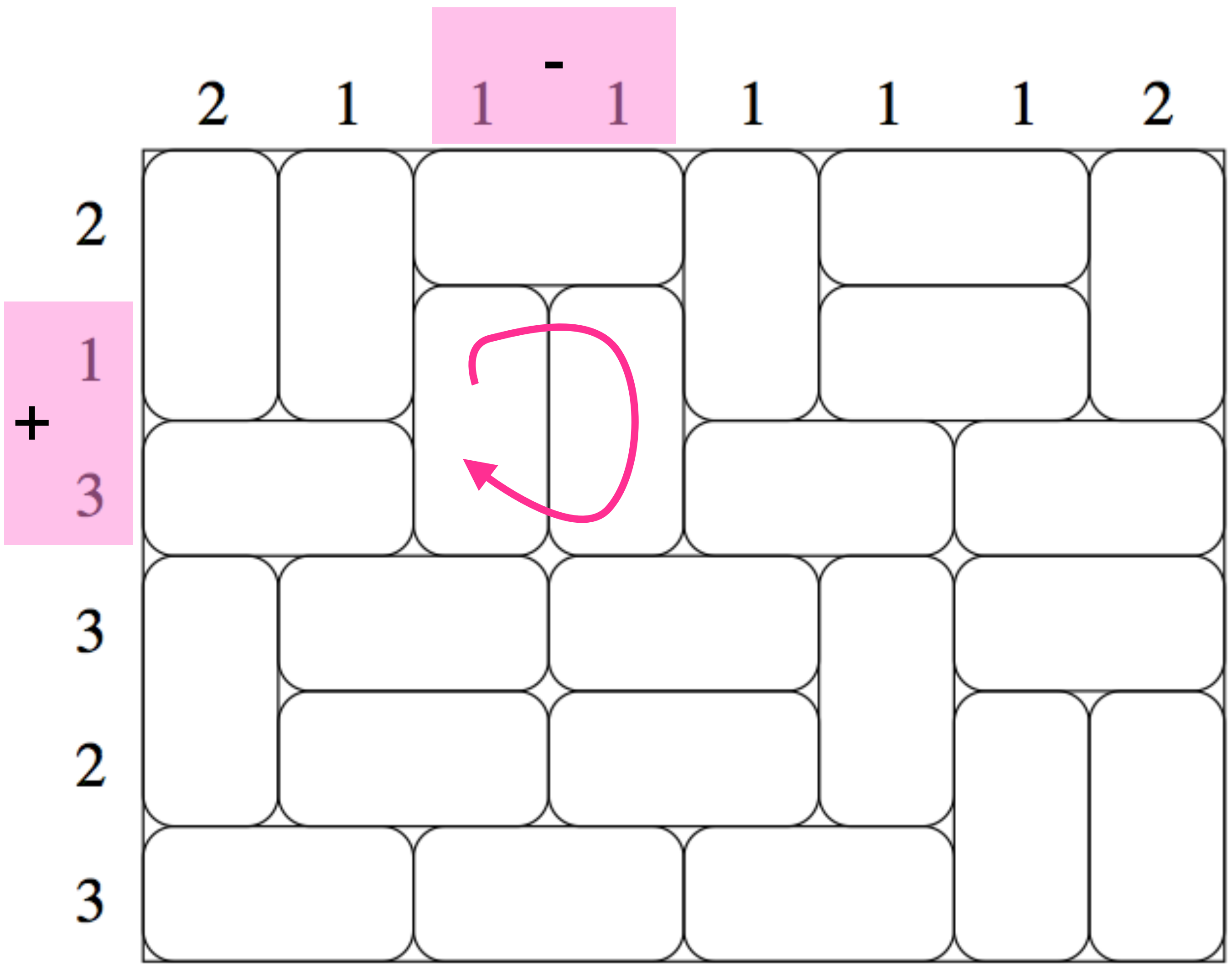
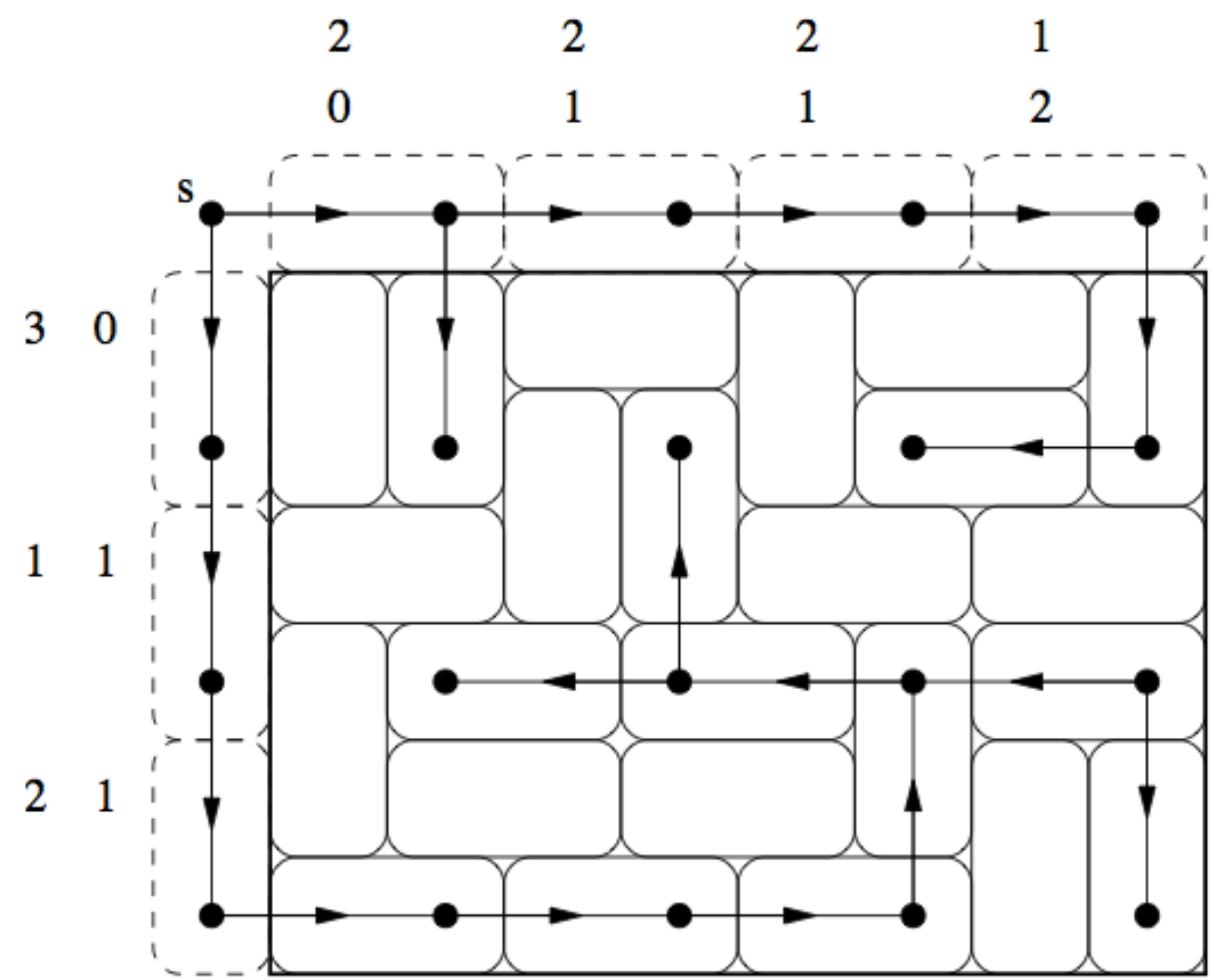


Figure: Nicolas Thiant

- [Problem] Ryser's theorem and the generalized algorithms heavily rely on a simple transformation of a tiling, which changes only column projections, while preserveing row projections.
- At a first glance we seem to miss this

Ingredient 1: Temperley's bijection

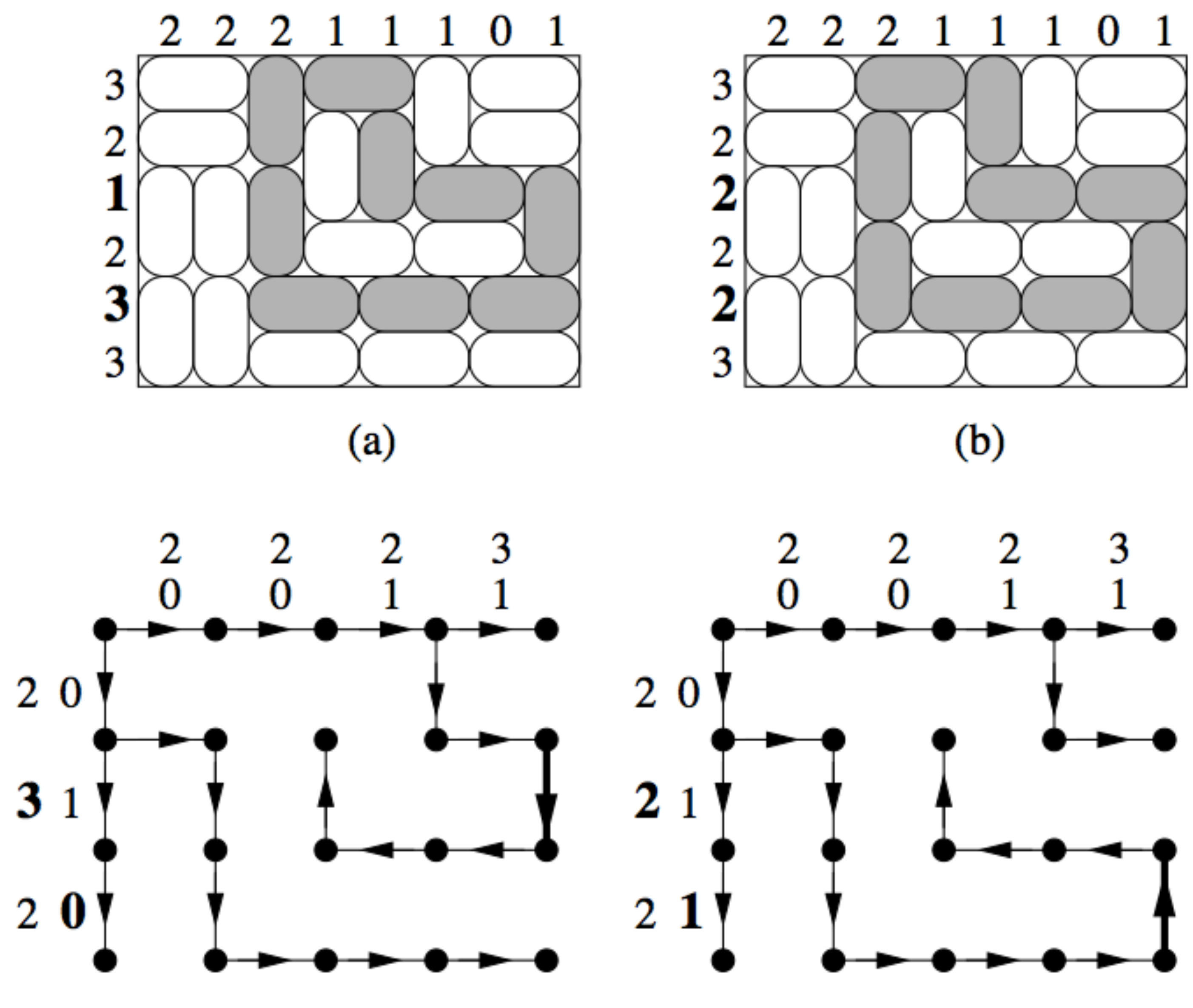
- [Wlog] we can assume that the grid to be tiled has odd*odd dimension minus upper left corner
- [Vertices] Consider cells of (odd,odd) coordinates.
- [Arcs] Covering domino defined directed connection with neighboring vertex
- [Spanning Tree] This is a directed spanning tree with root=upper left corner vertex
- [Thm] There is a bijection between domino tilings (and their projections) with directed spanning trees (and their projection)



Ingredient 2: Simple transformation

[the one we were looking for]

- Spanning trees can be transformed by changing origin of arc for some destination
- This changes only row projections
- [Algorithm] Start with a canonical tiling.
  Change until row projection are as required.
  Do the same for column projections.



(a) (b) (a') (b')

Open Problems

[the slide you were waiting for]

- [Fethi Jarray] place pebbles with given row and column projections such that for every pebble in (i,j), we never have (i,j-1) and (i,j+1) empty.
- [domino tilings] Define and optimize algorithm. Formulate a caracterization of projections which admit a solution.
- [prospective] relate methods from continuous math with methods from discrete math
- [practical] explore approximations for NP-hard tiling problems